Studies in Mathematical Sciences
Vol. 2, No. 1, 2011, pp. 73-79
www.cscanada.org

ISSN 1923-8444 [Print]
ISSN 1923-8452 [Online]
www.cscanada.net

Uniform Convergence, Mixing and Chaos

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Abstract: Let (X, d) be a compact metric space and let f_n: $X \to X$ be a sequence of continuous maps such that $\{f_n\}$ converges uniformly to a continuous and surjective map f: $X \to X$. We investigate the elements of topological mixing and the chaotic behavior possessed by f_n that can be inherited by f.

Key Words: Topological mixing; Weakly topological mixing; Distributional chaos in a sequence; Uniform convergence

1. INTRODUCTION

A dynamical system may be defined as a deterministic mathematical model for evolving the state of a system forwarded in time (the time here can be a continuous or discrete variable), which can be represented by a set of maps that specify how variables change over the time. It is well known that a number of real world problems can be modeled by a discrete dynamical system:

$$x_{n+1} = f(x_n),\ n = 0, 1, 2, \cdots$$

where $x_n \in X$, X is a compact metric space, and $f : X \to X$ is a continuous map.

Topological transitivity is a global characteristic of a dynamical system, the concept of which goes back to G. D. Birkhoff. According to [1], this concept was used by him in 1920 ([2], vol. 2, p. 108 and p. 221; see also [3]). Ruelle and Takens[4] considered a chaotic system a transitive system with the sensitive dependence on initial conditions. Li and Yorke[5] thought that a system is chaotic if there is an uncountable scrambled set in its domain. The chaotic system defined by Devaney[6] is a system which is chaotic in the sense of Ruelle and Takens with a dense set of periodic points. However, many authors (for example, Ref. [7]) found that a transitive system with a dense set of periodic points has to be of sensitive dependence on initial conditions. Huang and Ye[8] studied the transitive system with a fixed point, and they showed that such a system is chaotic in the sense of Li-Yorke. Their work led to solving an open problem,

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†Received 1 December 2010; accepted 24 December 2010.

i.e., whether or not Devaney's chaos implies the Li-Yorke chaos. The approach used by Huang and Ye is quite compact. Xiong and Yang[9], and Xiong and Chen[10] described the chaotic phenomena caused by topologically mixing maps, topologically weakly mixing maps or measurable mixing maps, using the words different from Li and Yorke's. Xiong used the approaches in the Refs. [9, 10] to discover the chaotic phenomena in a transitive system.

Topological transitivity is a necessary but not sufficient condition for the ergodicity of a dynamical system[11–13]. Often the proof of topological transitivity of a system was followed by the proof of its ergodicity. In connection with transitive points we wish to mention also the results of A. Iwanik. Two transitive points x and y in a standard dynamical system (X, f) (see [14, 15] for a more general situation) are called independent if (x, y) is transitive in the product system $(X, f)^2 = (X \times X, f \times f)$ (here $f \times f$ is defined by $(f \times f)(x, y) = (f(x), f(y))$). If two such points exist, then (X, f) is called topologically weakly mixing.

In [14] it is shown that, among other results, in every topologically weakly mixing system there exists an uncountable independent set. Totally independent sets have been studied in [15]. For the notion of "mixing" in the topological sense, which is analogous to the corresponding notion in ergodic theory, see also [16–19]. Here we just recall that topological strong mixing implies topological weak mixing, which implies topological transitivity.

In some recent papers[20–24], the chaotic behavior of a time invariant continuous map on a metric space has been extensively studied. Tian and Chen[22] introduced several new concepts and discussed the chaotic behavior of a time-varying map. Román-Flores[21] showed that if (X, d) is a metric space and $f_n : X \to X$ is a sequence of continuous maps such that $\{f_n\}$ converges uniformly to a function f and f_n is transitive for all natural numbers n, then f is not necessarily transitive. He gave sufficient conditions for the transitivity of the limit function f. Raghib and Kifah[23, 24] came to a conclusion that the inheritance of a classical property associated with chaos, such as topological transitivity, does not hold in general when taking uniform limits, not even in the particular case of continuous maps on a compact interval.

Several other concepts, such as chaos for a sequence of time invariant maps, have also been introduced by some authors. It is a well-known fact from calculus that if a sequence of continuous maps converges uniformly, then the limit map is continuous. Also, the limit map of a uniformly convergent sequence of Riemann-integrable maps is itself Riemann-integrable. However, differentiability of the limit map is not assured by uniform convergence of a sequence of differentiable maps. Therefore, it is worthwhile and of interest to investigate the properties of the elements of a uniformly convergent sequence of maps that can be inherited by the limit map. For this reason the limit map f of a uniformly convergent sequence of maps has been discussed extensively in [20–25].

In this paper, we use the method in [20] which investigates the conditions of topological transitivity of the uniform limit function f of a sequence of continuous topologically transitive maps (in strongly successive way). Let (X, d) be a compact metric space and suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous maps from X into X such that $f_n \to f$ uniformly, where f is a continuous map from X to X. If the sequence $\{f_n\}_{n=1}^{\infty}$ is topologically transitive on X in strongly successive way, then f can be shown to be topologically transitive. Based on this result, we further study and obtain a more general result, that is, the uniform limit function is also topologically mixing and weakly topologically mixing. Finally we deduce that it is distributively chaotic in a sequence and Li-Yorke chaotic.

Vellekoop[26] has already obtained that on intervals, transitivity implies chaos, so it is meaningless to continue discussing topological transitivity and chaos on intervals. Therefore, we study all the questions in the compact metric space in the paper. And we give an answer to the question: which factors of the complexity are presented by uniform convergence. We also use the definition of a sequence of topologically transitive maps in the strongly successive way given in [20] and obtain the following theorem.

Theorem 1.1 *Let (X, d) be a compact metric space and suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous maps from X into X such that $f_n \to f$ uniformly, where f is a continuous and surjective map from X to X.*

If the sequence $\{f_n\}_{n=1}^{\infty}$ is topologically transitive on X in the strongly successive way, then

(1) f is topologically mixing.

(2) f is weakly topologically mixing.

(3) f is topologically ergodic.

(4) f is distributively chaotic in a sequence.

(5) f is Li-York chaotic.

Devaney's chaos and Wiggins' chaos both include the concept of topological transitivity. And the Devaney chaos implies the Li-Yorke chaos. The chaos theory has widely been applied in various scientific fields including biology, information technology, economic science, engineering science, physics and demography etc. As an important chaotic behavior, transitivity also plays a significant role in practical applications.

2. PRELIMINARIES

In this section, some basic concepts and lemmas are introduced. This section is divided into two subsections.

2.1 Some Basic Concepts

Throughout this paper we always assume that (X, d) is a compact metric space. If A is an arbitrary set, then we denote the boundary of A by $B(A)$. The radius of any set A is denoted by $\text{Rad}(A)$. For an arbitrary $\varepsilon > 0$, we denote the ε-neighborhood of any point x by $N_\varepsilon(x)$.

Definition 2.1 *Let (X, d) be a compact metric space, $f_n : X \to X$ be a sequence of continuous maps defined on X, $n = 1, 2, \cdots$, and $f : X \to X$ be a continuous map. If for an arbitrary $\varepsilon > 0$, $d(f_n(x), f(x)) < \varepsilon$ for all $n \geq n_0$ and for all $x \in X$, where n_0 is a positive integer (depending on ε only), then we say that $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent to f on X. If $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent to f, we then write $f_n \to f$ uniformly.*

Definition 2.2 *Let (X, d) be a compact metric space, $f : X \to X$ be a continuous map, and U and V be any nonempty open subsets of X.*

(1) If there exists an integer $n > 0$ such that $f^n(U) \cap V \neq \emptyset$, then f is said to be topologically transitive.

(2) If $f \times f$ is transitive, i.e., for any nonempty open sets U_1, U_2, V_1 and V_2, there is an integer $n > 0$ such that $f^n(U_1) \cap V_1 \neq \emptyset$ and $f^n(U_2) \cap V_2 \neq \emptyset$, then f is said to be weakly topologically mixing.

(3) If there exists an integer $N > 0$ such that $f^n(U) \cap V \neq \emptyset$ for each $n \geq N$, then f is said to be topologically mixing.

It is clear that a mixing map is weakly mixing and a weakly mixing map is transitive.

Definition 2.3 *Let (X, d) be a compact metric space, $f : X \to X$ be a continuous map, and U and V be any two non-empty open subsets of X. If*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} Q(f^i(U) \cap V) = \lim_{n\to\infty} \frac{1}{n}\{\#(K(U, V) \cap \{0, 1, 2, \cdots, n-1\})\} > 0,$$

where $\#(\cdot)$ denotes the cardinality of a set,

$$Q(U) = \begin{cases} 1 & \text{if } U \neq \emptyset \\ 0 & \text{otherwise}, \end{cases}$$

and

$$K(U, V) = \{n \in N | f^n(U) \cap V \neq \emptyset\},$$

then we say that f is topologically ergodic.

Let (X, d) be a compact metric space and $\{p_i\}$ be an increasing sequence of positive integers. For any $x, y \in X$ and $t > 0$, let

$$F_{xy}(t, p_k) = \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{[0,t)}(d(f^{p_k}(x), f^{p_k}(y))),$$

$$F^*_{xy}(t, p_k) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{[0,t)}(d(f^{p_k}(x), f^{p_k}(y))),$$

where $\chi_A(y)$ is 1 if $y \in A$ and 0 otherwise. Obviously, F_{xy} and F^*_{xy} are both nondecreasing functions. If for $t \leq 0$ we define $F_{xy}(t) = F^*_{xy}(t) = 0$, then both F_{xy} and F^*_{xy} are probability distribution functions.

Definition 2.4 *Let $D \subset X$. If for all $x, y \in D$ with $x \neq y$, we have*

(1) $F_{xy}(\varepsilon, p_k) = 0$, for some $\varepsilon > 0$,

(2) $F^*_{xy}(t, p_k) = 1$, for any $t > 0$, then D is said to be a distributional chaotic set in a sequence. The two points x and y are said to display distributional chaos in a sequence. If f has a distributionally chaotic set which is uncountable in a sequence, f is said to be distributionally chaotic in a sequence.

Definition 2.5 *Let (X, d) be a compact metric space and $f : X \to X$ be continuous. $D \subset X$ is said to be a chaotic set of f if for any pair $(x, y) \in D \times D$, $x \neq y$, we have*

$$\liminf_{k\to\infty} d(f^k(x), f^k(y)) = 0,$$

$$\limsup_{k\to\infty} d(f^k(x), f^k(y)) > 0.$$

f is said to be chaotic in the sense of Li-Yorke, if it has a chaotic set D that is uncountable.

It is clear that chaos in a sequence must be Li-York chaotic.

Definition 2.6 *Let $f_n : X \to X$ be a sequence of continuous maps on a compact metric space X, $n = 1, 2, \cdots$. If for any two pairs of distinct non-empty open subsets U_1, V_1, and U_2, V_2 of X, there exist positive integers $K_1 \neq K_2$ such that $f_{K_1}(U_1) \cap V_1 \neq \emptyset$, $f_{K_2}(U_2) \cap V_2 \neq \emptyset$, then the sequence of maps $\{f_n\}_{n=1}^{\infty}$ is said to be topologically transitive on X in the strongly successive way.*

2.2 Some Lemmas

Lemma 2.7 *If f is topologically mixing, then f is topologically ergodic.*

See [27] for its proof.

Lemma 2.8 *Let (X, f) be a dynamical system relative to G, where X is a separable metric space containing at least two points and G is a semigroup of times. If f is topologically weakly mixing, then it is distributional chaos in a sequence.*

Proof. For a proof see [25].

Proof of the Theorem 1.1

(1) Step 1. We show that for any $U_1, V_1 \subset X$, if U_1, V_1 are open, then $f(U_1) \cap V_1 \neq \emptyset$.

Since the sequence $\{f_n\}_{n=1}^{\infty}$ is topologically transitive on X in the strongly successive way, there exists a positive integer k_1 such that $f_{k_1}(U_1) \cap V_1 \neq \emptyset$. Let $\varepsilon^{'}, \varepsilon^{''} > 0$. Consider an open ball $U_2 \subset U_1$ such that Rad(U_2)=$\varepsilon^{'}/2$ and the minimum distance of $B(U_2)$ from $B(U_1)$ is $\varepsilon^{'}/100$. Similarly we take an open ball $V_2 \subset V_1$ such that Rad(V_2)=$\varepsilon^{''}/2$ and the minimum distance of $B(V_2)$ from $B(V_1)$ is $\varepsilon^{''}/100$. Again by transitivity (in the strongly successive way), there exists a positive integer k_2 such that $f_{k_2}(U_2) \cap V_2 \neq \emptyset$. We now take an open ball $U_3 \subset U_2$ such that Rad$(U_3) = \varepsilon^{'}/3$ and the center of U_3 is the same as that of U_2. Similarly we take an open ball $V_3 \subset V_2$ such that Rad$(V_3) = \varepsilon^{''}/3$ and the center of V_3 is the same as that of V_2. Then by transitivity (in the strongly successive way) again, there exists a positive integer k_3 such that $f_{k_3}(U_3) \cap V_3 \neq \emptyset$. See Figure 1 and Figure 2, we now continue this process repeatedly. Then by our definition, $\{k_n\}_{n=1}^{\infty}$ is an infinite subset of all natural numbers. So, we rearrange this set as a sequence by taking the least element first then the next least element and so on. We now denote this rearrangement by $\{k_{n'}\}_{n=1}^{\infty}$. That is, $k_{1'}$ the least element of $\{k_n\}_{n=1}^{\infty}$.

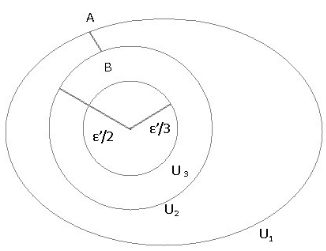

Figure 1: U_i is an open set of X and $U_{i+1} \subset U_i$, for all $i = 1, 2, \ldots$

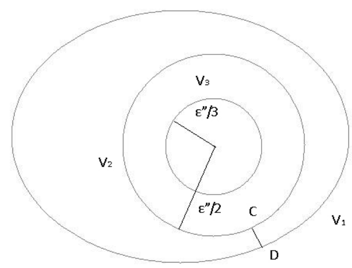

Figure 2: V_i is an open set of X and $V_{i+1} \subset V_i$, for all $i = 1, 2, \ldots$

By our construction $AB = \varepsilon^{'}/100$ and $CD = \varepsilon^{''}/100$. Also, $f_{k_1}(U_1) \cap V_1 \neq \emptyset$, $f_{k_2}(U_2) \cap V_2 \neq \emptyset$, $f_{k_3}(U_3) \cap V_3 \neq \emptyset$ and so on.

Then the following facts are noticeable:

a) U_i and V_i are open sets such that $U_{i+1} \subset U_i$ and $V_{i+1} \subset V_i$, for all $i = 1, 2, \cdots$.

b) There exists a sequence of positive integers $\{k_{n'}\}_{n=1}^{\infty}$ such that $f_{k_{n'}}(U_{n'}) \cap V_{n'} \neq \emptyset$, for all $n^{'}$.

c) U_i's (and V_i's) are all open sets such that the centers of U_i's (and V_i's) are the same for $i = 2, 3, \cdots$.

d) By a) and b), it can be easily proved that $f_{k_{n'}}(U_{n'}) \cap V_1 \neq \emptyset$, for all $k_{n'}$.

Obviously $f_{k_{n'}} \to f$ uniformly for $n = 2, 3, \cdots$, since $\{f_{k_{n'}}\}_{n=2}^{\infty}$ is a subsequence of $\{f_{k_{n'}}\}_{n=1}^{\infty}$. Then for $\varepsilon = \varepsilon^{''}/1000$, $d(f_{k_{n'}}(x), f(x)) < \varepsilon$, for all $n^{'} \geq m^{'}$ and for all $x \in X$, with some

$$m^{'} > 1. \tag{1}$$

We now show that $f(U_1) \cap V_1 \neq \emptyset$.

Let

$$y \in f_{k_{m'}}(U_{m'}) \cap V_{m'}. \tag{2}$$

Then $y \in V_{m'}$ and $y \in f_{k_{m'}}(U_{m'})$. So, there exists $x \in U_{m'}$ such that $f_{k_{m'}}(x) = y$. Again from (1) we get $d(f_{k_{m'}}(x), f(x)) < \varepsilon$.

So, $f(x) \in N_\varepsilon(f_{k_{m'}}(x))$, that is,

$$f(x) \in N_\varepsilon(y). \tag{3}$$

From (2) we get $y \in f_{k_{m'}}(U_1)$ and $y \in V_1$. Now $x \in U_{m'} \Rightarrow x \in U_1 \Rightarrow f(x) \in f(U_1)$. Since $m' \neq 1$, then by the definition of ε, (3) and also our construction above, we see that $f(x) \in V_1$. Hence, $f(U_1) \cap V_1 \neq \emptyset$.

Step 2. In order to prove that f is topologically mixing, we just need to prove that for any $n \geq 1$, $f^n(U_1) \cap V_1 \neq \emptyset$ is true.

Suppose that U_1 and V_1 are open sets in X. We see that the preimage $f^{-1}(V_1)$ of V_1 is open from the continuity of f. Therefore, for any $n \geq 1$, $f^{-n}(V_1) = f^{-1}(f^{-1}(\cdots(f^{-1}(V_1))))$ is an open subset in X.

From step 1, for any open set U_1 and V_1 in X, $f(U_1) \cap V_1 \neq \emptyset$ is true, so $f(U_1) \cap f^{-(n-1)}(V_1) \neq \emptyset$. Hence, $\emptyset \neq f^{n-1}[f(U_1) \cap f^{-(n-1)}(V_1)] \subset f^n(U_1 \cap V_1)$, that is, $f^n(U_1) \cap V_1 \neq \emptyset$. Therefore, f is topologically mixing.

(2) Since f is topologically mixing, it is topologically weakly mixing.

(3) By lemma 1, we can see that f is topologically ergodic.

(4) By lemma 2, we can see that f is distributively chaotic in a sequence.

(5) If f is distributively chaotic in a sequence, then f must be Li-Yorke chaotic. □

ACKNOWLEDGEMENTS

This work is supported by the major basic research fund of the Department of Education of Liaoning Province No. 2009A141 and the NSFC No. 10971245 and the Independent Fund of Central Universities No. 10010101.

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