CSCanada
Progress in Applied Mathematics
Vol. 6, No. 1, 2013, pp. [40–53]
DOI: 10.3968/j.pam.1925252820130601.2491

ISSN 1925-251X [Print]
ISSN 1925-2528 [Online]
www.cscanada.net
www.cscanada.org

Extended Matrix Variate Beta Distributions

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Supported by The Sistema Universitario de Investigación, Universidad de Antioquia by the project No. IN10164CE.

Received: May 4, 2013/ Accepted: July 11, 2013/ Published: July 31, 2013

Abstract: In this paper, we study the matrix variate generalization of the extended beta type 1 distribution. We also define extended matrix variate beta type 2 and type 3 distributions and derive several of their properties. We also establish relationship between these three matrix variate distributions.

Key words: Beta distribution; Beta function; Extended beta function; Extended matrix variate beta distribution; Matrix argument

Nagar, D. K., & Roldán-Correa, A. (2013). Extended Matrix Variate Beta Distributions. *Progress in Applied Mathematics, 6*(1), 40–53. Available from http://www.cscanada.net/index.php/pam/article/view/j.pam.1925252820130601.2491 DOI: 10.3968/j.pam.1925252820130601.2491

1. INTRODUCTION

A random variable u is said to have an extended beta type 1 distribution with parameters (p, q, σ), denoted by $u \sim \text{EB1}(p, q; \sigma)$, if its probability density function (p.d.f.) is given by (Chaudhry *et al.* [3], Nagar, Morán-Vásquez and Gupta [11]),

$$\frac{u^{p-1}(1-u)^{q-1}}{B(p,q;\sigma)} \exp\left[-\frac{\sigma}{u(1-u)}\right], \quad 0 < u < 1. \tag{1}$$

The extended beta function $B(a, b; \sigma)$ used above is defined as

$$B(a,b;\sigma) = \int_0^1 t^{a-1}(1-t)^{b-1} \exp\left[-\frac{\sigma}{t(1-t)}\right] \mathrm{d}t, \tag{2}$$

where a and b are arbitrary complex numbers and $\mathrm{Re}(\sigma) > 0$. If $\sigma = 0$, then $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(b) > 0$. For $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(b) > 0$, it is clear that $B(a, b, 0) = B(a, b)$. The rational and justification for introducing this function are given in Chaudhry *et al.* [3] where several properties and a statistical application have also been studied. Miller [8] further studied this function and has given several additional results. Recently, Morán-Vásquez and Nagar [9] have applied the extended beta function in deriving certain probability distributions. The extended beta type 1 distribution can be used in Bayesian methodology as a prior distribution on the success probability of a binomial distribution.

A random variable v is said to have an extended beta type 2 distribution with parameters (p, q, σ), denoted by $v \sim \mathrm{EB2}(p, q; \sigma)$, if its p.d.f. is given by

$$\frac{v^{p-1}(1+v)^{-(p+q)}}{B(p, q; \sigma)\exp(2\sigma)} \exp\left[-\sigma\left(v + \frac{1}{v}\right)\right], \quad v > 0. \tag{3}$$

Since (3) can be obtained from (1) by the transformation $v = u/(1-u)$ the distribution of v can also be called the *inverted extended beta distribution*. By using the transformation $w = u/(2-u)$, the extended beta type 3 density is obtained as (Gupta and Nagar [5,6], Cardeño, Nagar and Sánchez [2], Nagar and Ramirez-Vanegas [12,13]),

$$\frac{2^p w^{p-1}(1-w)^{q-1}}{B(p, q; \sigma)(1+w)^{p+q}} \exp\left[-\frac{\sigma(1+w)^2}{2w(1-w)}\right], \quad 0 < u < 1. \tag{4}$$

For $\sigma = 0$ with $p > 0$ and $q > 0$, the extended beta type 1, type 2 and type 3 distributions reduce to standard beta type 1, type 2 and type 3 distributions, respectively. The beta type 1, type 2 and type 3 distributions have been generalized to the matrix case in various ways. These generalizations and some of their properties can be found in Olkin and Rubin [15], Gupta and Nagar [4–6], and Muirhead [10]. For some recent advances the reader is referred to Hassairi and Regaig [7], Ben-Farah and Hassairi [1], and Zine [16]. However, generalizations of extended beta distributions to the matrix case have not been studied.

In this article, we consider matrix variate generalizations of extended beta type 1, extended beta type 2 and extended beta type 3 distributions defined by the densities (1), (3) and (4), respectively. We derive several properties of these distributions including joint probability density functions of the eigenvalues.

2. SOME DEFINITIONS AND PRELIMINARY RESULTS

In this section we give some definitions and preliminary results which are used in subsequent sections.

We begin with a brief review of some definitions and notations. We adhere to standard notations (cf. Gupta and Nagar [4]). Let $A = (a_{ij})$ be an $m \times m$ matrix. Then, A' denotes the transpose of A; $\mathrm{tr}(A) = a_{11} + \cdots + a_{mm}$; $\mathrm{etr}(A) = \exp(\mathrm{tr}(A))$; $\det(A)$ = determinant of A; $A \geq 0$ means that A is symmetric positive semi-definite; $A > 0$ means that A is symmetric positive definite and $A^{1/2}$ denotes the unique symmetric positive definite square root of $A > 0$. The multivariate gamma function is defined by

$$\Gamma_m(a) = \int_{X>0} \mathrm{etr}(-X) \det(X)^{a-(m+1)/2}\, \mathrm{d}X$$

$$= \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma\left(a - \frac{i-1}{2}\right), \quad \mathrm{Re}(a) > \frac{m-1}{2}. \tag{5}$$

The multivariate generalization of the beta function is given by

$$\begin{aligned} B_m(a,b) &= \int_0^{I_m} \det(X)^{a-(m+1)/2} \det(I_m - X)^{b-(m+1)/2} \, \mathrm{d}X \\ &= \frac{\Gamma_m(a)\Gamma_m(b)}{\Gamma_m(a+b)} = B_m(b,a), \quad \mathrm{Re}(a) > \frac{m-1}{2}, \quad \mathrm{Re}(b) > \frac{m-1}{2}. \end{aligned} \tag{6}$$

Definition 2.1. *The extended matrix variate beta function, denoted by* $B_m(a,b;\Sigma)$*, is defined as*

$$\begin{aligned} B_m(a,b;\Sigma) = \int_0^{I_m} & \mathrm{etr}[-\Sigma Z^{-1}(I_m - Z)^{-1}] \\ & \times \det(Z)^{a-(m+1)/2} \det(I_m - Z)^{b-(m+1)/2} \, \mathrm{d}Z, \end{aligned} \tag{7}$$

where a *and* b *are arbitrary complex numbers and* $\mathrm{Re}(\Sigma) > 0$*. If* $\Sigma = 0$*, then* $\mathrm{Re}(a) > (m-1)/2$ *and* $\mathrm{Re}(b) > (m-1)/2$*.*

From (7) it is clear that $B_m(a,b;\Sigma) = B_m(b,a;\Sigma)$. Further, in the above definition if we take $\Sigma = 0$, then for $\mathrm{Re}(a) > (m-1)/2$, $\mathrm{Re}(b) > (m-1)/2$, we have $B_m(a,b;0) = B_m(a,b)$.

Theorem 2.1. *For* a *and* b *arbitrary complex numbers and* $\mathrm{Re}(\Sigma) > 0$*,*

$$B_m(a,b;\Sigma) = \mathrm{etr}(-2\Sigma) \int_{U>0} \frac{\mathrm{etr}[-\Sigma(U + U^{-1})] \det(U)^{a-(m+1)/2}}{\det(I_m + U)^{a+b}} \, \mathrm{d}U. \tag{8}$$

Further, the above result also holds good for $\Sigma = 0$ *if* $\mathrm{Re}(a) > (m-1)/2$ *and* $\mathrm{Re}(b) > (m-1)/2$*.*

Proof. Making the substitution $Z = (I_m + U)^{-1}U$ with the Jacobian $J(Z \to U) = \det(I_m + U)^{-(m+1)}$ in (7), we get the desired result. □

The extended matrix variate beta function has been defined and studied recently by Nagar, Roldán-Correa and Gupta [14].

3. THE DENSITY FUNCTIONS

Recently, Nagar, Roldán-Correa and Gupta [14] have defined a matrix variate generalization of the extended beta type 1 distribution as follows:

Definition 3.1. *An* $m \times m$ *random positive definite matrix* U *is said to have an extended matrix variate beta type 1 distribution with parameters* (p,q,Σ)*, denoted as* $U \sim \mathrm{EB1}(m,p,q;\Sigma)$*, if its p.d.f. is given by*

$$\frac{\mathrm{etr}[-\Sigma U^{-1}(I_m - U)^{-1}] \det(U)^{p-(m+1)/2} \det(I_m - U)^{q-(m+1)/2}}{B_m(p,q;\Sigma)}, \tag{9}$$

where $0 < U < I_m$*,* $-\infty < p < \infty$*,* $-\infty < q < \infty$ *and* $\Sigma > 0$*.*

From the definition it is clear that if $U \sim \mathrm{EB1}(m, p, q; \Sigma)$, then $I_m - U \sim \mathrm{EB1}(m, q, p; \Sigma)$.

We now proceed to define the extended matrix variate beta type 2 distribution.

Definition 3.2. *An $m \times m$ random positive definite matrix V is said to have an extended matrix variate beta type 2 distribution with parameters (p, q, Σ), denoted as $V \sim \mathrm{EB2}(m, p, q; \Sigma)$, if its p.d.f. is given by*

$$\frac{\mathrm{etr}[-\Sigma(V + V^{-1})] \det(V)^{p-(m+1)/2} \det(I_m + V)^{-(p+q)}}{B_m(p, q; \Sigma)\, \mathrm{etr}(2\Sigma)}, \tag{10}$$

where $V > 0$, $-\infty < p < \infty$, $-\infty < q < \infty$ and $\Sigma > 0$.

The density (10) can be obtained from (9) by the transformation $U = (I_m + V)^{-1}V$, with the Jacobian $J(U \to V) = \det(I_m + V)^{-(m+1)}$. Since the matrix variate beta type 2 distribution is also known as the matrix variate F-distribution, extended matrix variate beta type 2 distribution can be also be called extended matrix variate F-distribution.

Note that in Definition 3.1 and Definition 3.2 if we take $\Sigma = 0$, $p > (m-1)/2$ and $q > (m-1)/2$, then (9) and (10) slide to matrix variate beta type 1 and matrix variate beta type 2 densities given by

$$\frac{\det(U)^{p-(m+1)/2} \det(I_m - U)^{q-(m+1)/2}}{B_m(p, q)}, \quad 0 < U < I_m \tag{11}$$

and

$$\frac{\det(V)^{p-(m+1)/2} \det(I_m + V)^{-(p+q)}}{B_m(p, q)}, \quad V > 0, \tag{12}$$

respectively.

As we will see in the following theorem, using a linear transformation on the matrix U, we can generalize the extended matrix variate beta type 1 distribution.

Theorem 3.1. *Let $U \sim \mathrm{EB1}(m, p, q; \Sigma)$, and Ψ and Ω be two constant matrices of order m such that $\Omega > 0$, $\Psi \geq 0$ and $\Omega - \Psi > 0$. Then, the $m \times m$ random matrix X defined by*

$$X = (\Omega - \Psi)^{1/2} U (\Omega - \Psi)^{1/2} + \Psi \tag{13}$$

has the p.d.f. given by

$$\frac{\det(X - \Psi)^{p-(m+1)/2} \det(\Omega - X)^{q-(m+1)/2}}{B_m(p, q; \Sigma) \det(\Omega - \Psi)^{p+q-(m+1)/2}} \times \mathrm{etr}[-\Sigma(\Omega - \Psi)^{1/2}(X - \Psi)^{-1}(\Omega - \Psi)(\Omega - X)^{-1}(\Omega - \Psi)^{1/2}], \tag{14}$$

where $\Psi < X < \Omega$.

Proof. The Jacobian of the transformation (13) is $J(U \to X) = \det(\Omega - \Psi)^{-(m+1)/2}$. Thus, the density of X is derived from the density of U by making appropriate substitutions. □

Definition 3.3. *An $m \times m$ random positive definite matrix X is said to have a generalized extended matrix variate beta type 1 distribution with parameters p, q, Σ, Ω and Ψ, denoted by $X \sim \mathrm{GEB1}(m, p, q; \Sigma, \Omega, \Psi)$, if its p.d.f. is given by (14).*

If we take$\Psi = 0$ and $\Omega = I_m$ in (14), then we obtain an extended matrix variate beta type 1 density. Moreover, from Theorem 3.1, it is clear that if $X \sim \mathrm{GEB1}(m, p, q; \Sigma, \Omega, \Psi)$ then $(\Omega - \Psi)^{-1/2}(X - \Psi)(\Omega - \Psi)^{-1/2} \sim \mathrm{EB1}(m, p, q; \Sigma)$.

Similarly, using linear transformation on the matrix V, we can generalize the extended matrix variate beta type 2 distribution.

Theorem 3.2. *Let $V \sim \mathrm{EB2}(m, p, q; \Sigma)$, and Ψ and Ω be two $m \times m$ constant symmetric matrices such that $\Omega > 0$ and $\Psi \geq 0$. Then, the $m \times m$ random matrix Y defined by*

$$Y = (\Omega + \Psi)^{1/2} V (\Omega + \Psi)^{1/2} + \Psi \tag{15}$$

has the p.d.f. given by

$$\begin{aligned}&\frac{\det(\Omega + \Psi)^q \det(Y - \Psi)^{p-(m+1)/2}}{B_m(p, q; \Sigma) \operatorname{etr}(2\Sigma) \det(\Omega + Y)^{p+q}} \operatorname{etr}[-\Sigma(\Omega + \Psi)^{-1/2}(Y - \Psi)(\Omega + \Psi)^{-1/2}] \\ &\quad \times \operatorname{etr}[-\Sigma(\Omega + \Psi)^{1/2}(Y - \Psi)^{-1}(\Omega + \Psi)^{1/2}], \quad Y > \Psi.\end{aligned} \tag{16}$$

Proof. The Jacobian of the transformation (15) is $J(V \to Y) = \det(\Omega + \Psi)^{-(m+1)/2}$. Now, by substituting appropriately the density of Y is derived. □

Definition 3.4. *An $m \times m$ random positive definite matrix Y is said to have a generalized extended matrix variate beta type 2 distribution with parameters p, q, Σ, Ω and Ψ, denoted by $Y \sim \mathrm{GEB2}(m, p, q; \Sigma, \Omega, \Psi)$, if its p.d.f. is given by (16).*

If we take $\Psi = 0$ and $\Omega = I_m$ in (16), then we obtain an extended matrix variate beta type 2 distribution. Moreover, from Theorem 3.2, it is clear that if $Y \sim \mathrm{GEB2}(m, p, q; \Sigma, \Omega, \Psi)$, then $(\Omega+\Psi)^{-1/2}(Y-\Psi)(\Omega+\Psi)^{-1/2} \sim \mathrm{EB2}(m, p, q; \Sigma)$.

4. PROPERTIES

In this section we give some properties of random matrices which are distributed as extended matrix variate beta type 1 and type 2.

Theorem 4.1. *Let $U \sim \mathrm{EB1}(m, p, q; \Sigma)$, and A be an $m \times m$ constant nonsingular matrix. Then, the p.d.f. of $X = AUA'$ is given by*

$$\frac{\det(X)^{p-(m+1)/2} \det(AA' - X)^{q-(m+1)/2} \operatorname{etr}[-\Sigma A' X^{-1} AA' (AA' - X)^{-1} A]}{B_m(p, q; \Sigma) \det(AA')^{p+q-(m+1)/2}}, \tag{17}$$

where $0 < X < AA'$.

Proof. Transforming $X = AUA'$ with the Jacobian $J(U \to X) = \det(AA')^{-(m+1)/2}$ in the density of U, we get the desired result. □

Corollary 4.1.1. *Let $U \sim \mathrm{EB1}(m, p, q; \Sigma)$, and A be an $m \times m$ constant nonsingular symmetric matrix. Then, $AUA \sim \mathrm{GEB1}(m, p, q; \Sigma, A^2, 0)$.*

Proof. Replacing A' by A in (17), we get the result. □

Theorem 4.2. *Let $V \sim \mathrm{EB2}(m, p, q; \Sigma)$, and A be an $m \times m$ constant nonsingular matrix. Then, the p.d.f. of $Y = AVA'$ is given by*

$$\frac{\det(AA')^q \det(Y)^{p-(m+1)/2} \operatorname{etr}[-\Sigma(A^{-1} Y (A')^{-1} + A' Y^{-1} A)]}{B_m(p, q; \Sigma) \operatorname{etr}(2\Sigma) \det(AA' + Y)^{p+q}}, \quad Y > 0. \tag{18}$$

Proof. Transforming $Y = AVA'$, with the Jacobian $J(V \to Y) = \det(AA')^{-(m+1)/2}$, in the density of V, we get the desired result. □

Corollary 4.2.1. *Let $V \sim \text{EB2}(m, p, q; \Sigma)$, and A be an $m \times m$ constant nonsingular symmetric matrix. Then, $AVA \sim \text{GEB2}(m, p, q; \Sigma, A^2, 0)$.*

Proof. Replacing A' by A in (18) we get the result. □

In the following two theorems, we show that extended matrix variate beta distributions are orthogonally invariant when Σ is proportional to an identity matrix.

Theorem 4.3. *Let $U \sim \text{EB1}(m, p, q; \lambda I_m)$, $\lambda > 0$, and H be an $m \times m$ orthogonal matrix whose elements are constants or random variables distributed independently of U. If H is a constant matrix, then the distribution of U is invariant under the transformation $U \to HUH'$. Further, if H is random, then HUH' and H are independent, $HUH' \sim \text{EB1}(m, p, q; \lambda I_m)$.*

Proof. First, let H be a constant matrix. Then, from Theorem 4.1, $HUH' \sim \text{EB1}(m, p, q; \lambda I_m)$. Further, if H is a random orthogonal matrix, then $HUH'|H \sim \text{EB1}(m, p, q; \lambda I_m)$ and since this distribution does not depend on H, $HUH' \sim \text{EB1}(m, p, q; \lambda I_m)$. □

Theorem 4.4. *Let $V \sim \text{EB2}(m, p, q; \lambda I_m)$, $\lambda > 0$, and H be an $m \times m$ orthogonal matrix whose elements are constants or random variables distributed independently of V. If H is a constant matrix, then the distribution of V is invariant under the transformation $V \to HVH'$. Further, if H is random, then HVH' and H are independent, $HVH' \sim \text{EB2}(m, p, q; \lambda I_m)$.*

Proof. Similar to the proof of Theorem 4.3. □

Now, we exhibit the relationship between extended matrix variate beta type 1 and type 2 random matrices. First, we derive the densities of U^{-1} and V^{-1}.

Theorem 4.5. *If $U \sim \text{EB1}(m, p, q; \Sigma)$, then the p.d.f. of $X = U^{-1}$ is given by*

$$\frac{\text{etr}[-\Sigma X^2 (X - I_m)^{-1}] \det(X)^{-(p+q)} \det(X - I_m)^{q-(m+1)/2}}{B_m(p, q; \Sigma)}, \quad X > I_m, \tag{19}$$

Proof. Making the transformation $X = U^{-1}$ with the Jacobian $J(U \to X) = \det(X)^{-(m+1)}$ in (9), the density of X is obtained. □

The density (19) may be called the inverse extended matrix variate beta type 1. From Theorem 4.5, it is clear that if $U \sim \text{EB1}(m, p, q; \Sigma)$, then U^{-1} does not follow an extended matrix variate beta type 1 distribution. However, it can easily be observed that $X - I_m \sim \text{EB2}(m, q, p; \Sigma)$, that is, $U^{-1} - I_m \sim \text{EB2}(m, q, p; \Sigma)$. On the other hand, if a random matrix V has an extended matrix variate beta type 2 distribution, then the distribution of V^{-1} is also extended matrix variate beta type 2 as we will see in the following theorem.

Theorem 4.6. *If $V \sim \text{EB2}(m, p, q; \Sigma)$, then $Y = V^{-1} \sim \text{EB2}(m, q, p; \Sigma)$.*

Proof. Transforming $Y = V^{-1}$, with the Jacobian $J(V \to Y) = \det(Y)^{-(m+1)}$, in the density of V the desired result is obtained. □

Theorem 4.7. *If $U \sim \mathrm{EB1}(m,p,q;\Sigma)$ and $Y = (I_m - U)^{-1/2} U (I_m - U)^{-1/2}$, then $Y \sim \mathrm{EB2}(m,p,q;\Sigma)$. Further, if $V \sim \mathrm{EB2}(m,p,q;\Sigma)$ and $X = (I_m + V)^{-1/2} V (I_m + V)^{-1/2}$, then $X \sim \mathrm{EB1}(m,p,q;\Sigma)$.*

Proof. Since the matrix U commutes with any rational function of U, we can write $Y = (I_m - U)^{-1/2}\, U (I_m - U)^{-1/2} = (I_m - U)^{-1} U$ and the Jacobian of this transformation is $J(U \to Y) = \det(I_m + Y)^{-(m+1)}$. Now, making these substitutions, we get part one. For the second part, making the transformation $X = (I_m + V)^{-1/2} V (I_m + V)^{-1/2} = (I_m + V)^{-1} V$ with the Jacobian $J(V \to X) = \det(I_m - X)^{-(m+1)}$, we obtain the result. □

In the following theorem we compute expected values of functions of matrices distributed as extended beta type 1 and 2.

Theorem 4.8. *If $U \sim \mathrm{EB1}(m,p,q;\Sigma)$, then*

$$E[\det(U)^r \det(I_m - U)^s] = \frac{B_m(p+r,q+s;\Sigma)}{B_m(p,q;\Sigma)}.$$

Proof. By definition

$$\begin{aligned} E[\det(U)^r \det(I_m - U)^s] &= \frac{1}{B_m(p,q;\Sigma)} \int_0^{I_m} \mathrm{etr}[-\Sigma U^{-1}(I_m - U)^{-1}] \\ &\quad \times \det(U)^{p+r-(m+1)/2} \det(I_m - U)^{q+s-(m+1)/2}\, \mathrm{d}U \\ &= \frac{B_m(p+r,q+s;\Sigma)}{B_m(p,q;\Sigma)}, \end{aligned}$$

where the last line has been obtained using (7). □

Theorem 4.9. *If $V \sim \mathrm{EB2}(m,p,q;\Sigma)$, then*

$$E[\det(V)^r \det(I_m + V)^{-s}] = \frac{B_m(p+r,q+s-r;\Sigma)}{B_m(p,q;\Sigma)}.$$

Proof. By definition

$$\begin{aligned} E[\det(V)^r \det(I_m + V)^{-s}] &= \frac{1}{B_m(p,q;\Sigma)\,\mathrm{etr}(2\Sigma)} \int_{V>0} \mathrm{etr}[-\Sigma(V + V^{-1})] \\ &\quad \times \det(V)^{p+r-(m+1)/2} \det(I_m + V)^{-(p+q+s)}\, \mathrm{d}V \\ &= \frac{B_m(p+r,q+s-r;\Sigma)}{B_m(p,q;\Sigma)}, \end{aligned}$$

where the last line has been obtained by using (8). □

5. EXTENDED MATRIX VARIATE BETA TYPE 3 DISTRIBUTION

In this section we define the matrix variate beta type 3 distribution and derive several of its properties.

Definition 5.1. *An $m \times m$ random positive definite matrix W is said to have an extended matrix variate beta type 3 distribution with parameters (p, q, Σ), denoted as $W \sim \mathrm{EB3}(m, p, q; \Sigma)$, if its p.d.f. is given by*

$$\frac{2^{pm} \det(W)^{p-(m+1)/2} \det(I_m - W)^{q-(m+1)/2}}{B_m(p, q; \Sigma) \det(I_m + W)^{p+q}} \times \mathrm{etr}\left[-\frac{1}{2}\Sigma W^{-1}(I_m - W)^{-1}(I_m + W)^2\right], \tag{20}$$

where $0 < W < I_m$, $-\infty < p < \infty$, $-\infty < q < \infty$ and $\Sigma > 0$.

Note that in the Definition 5.1 if we take $\Sigma = 0$, $p > (m-1)/2$ and $q > (m-1)/2$, then (20) slides to a matrix variate beta type 3 density given by

$$\frac{2^{pm} \det(W)^{p-(m+1)/2} \det(I_m - W)^{q-(m+1)/2}}{B_m(p, q) \det(I_m + W)^{p+q}}, \quad 0 < W < I_m.$$

Theorem 5.1. *Let $W \sim \mathrm{EB3}(m, p, q; \Sigma)$, and A be an $m \times m$ constant nonsingular matrix. Then, the p.d.f. of $X = AWA'$ is given by*

$$\frac{2^{mp} \det(X)^{p-(m+1)/2} \det(AA' - X)^{q-(m+1)/2}}{B_m(p, q; \Sigma) \det(AA')^{-(m+1)/2}} \times \mathrm{etr}\left[-\frac{1}{2}\Sigma A'X^{-1}AA'(AA' - X)^{-1}(AA' + X)(AA')^{-1}(AA' + X)(A^{-1})'\right],$$

where $0 < X < AA'$.

Proof. Similar to the proof of Theorem 4.1. □

In the following theorem, we show that extended matrix variate beta type 3 distribution is orthogonally invariant when Σ is proportional to an identity matrix.

Theorem 5.2. *Let $W \sim \mathrm{EB3}(m, p, q; \lambda I_m)$, $\lambda > 0$, and H be an $m \times m$ orthogonal matrix whose elements are constants or random variables distributed independently of W. If H is a constant matrix, then the distribution of W is invariant under the transformation $W \to HWH'$. Further, if H is random, then HWH' and H are independent, $HWH' \sim \mathrm{EB3}(m, p, q; \lambda I_m)$.*

Proof. Similar to the proof of Theorem 4.3. □

Theorem 5.3. *If $U \sim \mathrm{B1}(m, p, q; \Sigma)$, then $(I_m + U)^{-1}(I_m - U) \sim \mathrm{B3}(m, q, p; \Sigma)$ and $(2I_m - U)^{-1}U \sim \mathrm{B3}(m, p, q; \Sigma)$.*

Proof. In the p.d.f. (9) of U making the transformation $W = (I_p + U)^{-1}(I_p - U)$ with the Jacobian $J(U \to W) = 2^{m(m+1)/2}(I_m + W)^{-(m+1)}$, we get the desired result. The second part follows from the first part by noting that $(2I_m - U)^{-1}U = [I_m + (I_m - U)]^{-1}[I_m - (I_m - U)]$ and $I_m - U \sim \mathrm{B1}(m, q, p; \Sigma)$. □

Theorem 5.4. *If $V \sim \mathrm{B2}(m, p, q; \Sigma)$, then $(2I_m + V)^{-1}V \sim \mathrm{B3}(m, p, q; \Sigma)$ and $(I_m + 2V)^{-1} \sim \mathrm{B3}(m, q, p; \Sigma)$.*

Proof. Transforming $W = (2I_m + V)^{-1}V$ with the Jacobian $J(V \to W) = 2^{m(m+1)/2} (I_m - W)^{-(m+1)}$ in the p.d.f. (10) of V, we get the desired result. The second part follows from the first part by noting that $(I_m + 2V)^{-1} = (2I_m + V^{-1})^{-1}V^{-1}$ and $V^{-1} \sim \text{B2}(m, q, p; \Sigma)$. □

Finally, from Theorem 5.3 and Theorem 5.4, we get the following result.

Theorem 5.5. *If* $W \sim \text{B3}(m, p, q; \Sigma)$*, then* $2(I_m + W)^{-1}W \sim \text{B1}(m, p, q; \Sigma)$*,* $(I_m + W)^{-1}(I_m - W) \sim \text{B1}(m, q, p; \Sigma)$*,* $2(I_m - W)^{-1}W \sim \text{B2}(m, p, q; \Sigma)$ *and* $(I_m - W)W^{-1}/2 \sim \text{B2}(m, q, p; \Sigma)$*.*

6. EIGENVALUES OF EXTENDED BETA MATRICES

In this section, we derive densities of eigenvalues of random matrices distributed as extended matrix variate beta type 1, type 2 and type 3. First we state the following result which is useful in deriving main results of this section.

Theorem 6.1. *Let A be a positive definite random matrix of order m with the probability density function $f(A)$. Then, the joint p.d.f. of eigenvalues $l_1, l_2, \ldots, l_m$ of A is given by*

$$\frac{\pi^{m^2/2}}{\Gamma_m(m/2)} \prod_{i<j}^{m} (l_i - l_j) \int_{O(m)} f(HLH')[\mathrm{d}H], \qquad (l_1 > l_2 > \cdots > l_m > 0), \tag{21}$$

where $L = \operatorname{diag}(l_1, l_2, \ldots, l_m)$ and $[\mathrm{d}H]$ is the unit invariant Haar measure on the group of orthogonal matrices.

An important integral involving the invariant Haar measure on the group of orthogonal matrices is given by

$$\int_{O(m)} \operatorname{etr}(AHBH')\,[\mathrm{d}H] = {}_0F_0^{(m)}(A, B), \tag{22}$$

where A and B are symmetric matrices of order m and ${}_0F_0^{(m)}(A, B)$ is the hypergeometric function of two matrix arguments. The function ${}_0F_0^{(m)}(A, B)$ is defined in terms of zonal polynomials as

$${}_0F_0^{(m)}(A, B) = \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{C_\kappa(A) C_\kappa(B)}{C_\kappa(I_m) k!},$$

where $\sum_{\kappa \vdash k}$ denotes summation over all ordered partitions κ, $\kappa = (k_1, \ldots, k_m)$, $k_1 \geq \cdots \geq k_m \geq 0$ and $k_1 + \cdots + k_m = k$; $C_\kappa(A)$, $C_\kappa(B)$ and $C_\kappa(I_m)$ are the zonal polynomials of A, B and I_m corresponding to the ordered partition κ.

Also, if one of the argument matrices is proportional to the identity matrix the function ${}_0F_0^{(m)}(A, B)$ reduces to a one argument function. That is, if $A = \alpha I_m$, then

$${}_0F_0^{(m)}(\alpha I_m, B) = {}_0F_0^{(m)}(\alpha B) = \operatorname{etr}(\alpha B).$$

Proof of Theorem 6.1 and several other results such as (22) can be found in Muirhead [10].

Theorem 6.2. *If* $U \sim \mathrm{EB1}(m,p,q;\Sigma)$*, then the joint p.d.f. of the eigenvalues* $u_1, u_2, \ldots, u_m$ *of* U *is given by*

$$\frac{\pi^{m^2/2}}{\Gamma_m(m/2)B_m(p,q;\Sigma)}\left[\prod_{i<j}^{m}(u_i-u_j)\right]\prod_{i=1}^{m}\left[u_i^{p-(m+1)/2}(1-u_i)^{q-(m+1)/2}\right]$$
$$\times\,{}_0F_0^{(m)}(-\Sigma, L^{-1}(1-L)^{-1}), \quad 0<u_m<\cdots<u_1<1, \tag{23}$$

where $L = \mathrm{diag}(u_1,\ldots,u_m)$.

Proof. The probability density function of U is given by (9). Applying Theorem 6.1, we obtain the joint p.d.f. of the eigenvalues $u_1, u_2, \ldots, u_m$ of U as

$$\frac{\pi^{m^2/2}}{\Gamma_m(m/2)B_m(p,q;\Sigma)}\left[\prod_{i<j}^{m}(u_i-u_j)\right]\prod_{i=1}^{m}\left[u_i^{p-(m+1)/2}(1-u_i)^{q-(m+1)/2}\right]$$
$$\times\int_{O(m)}\mathrm{etr}[-\Sigma H L^{-1}(I_m-L)^{-1}H'][\mathrm{d}H]. \tag{24}$$

Further, using (22), we have

$$\int_{O(m)}\mathrm{etr}[-\Sigma H L^{-1}(I_m-L)^{-1}H'][\mathrm{d}H] = {}_0F_0^{(m)}(-\Sigma, L^{-1}(I_m-L)^{-1}). \tag{25}$$

Finally substituting (25) in (24), we obtain the desired result. □

Corollary 6.2.1. *If* $U \sim \mathrm{EB1}(m,p,q;\lambda I_m)$*, then the joint p.d.f. of the eigenvalues* $u_1, u_2, \ldots, u_m$ *of* U *is given by*

$$\frac{\pi^{m^2/2}}{\Gamma_m(m/2)B_m(p,q;\lambda I_m)}\left[\prod_{i<j}^{m}(u_i-u_j)\right]\prod_{i=1}^{m}\left[u_i^{p-(m+1)/2}(1-u_i)^{q-(m+1)/2}\right]$$
$$\times\exp\left[-\lambda\sum_{i=1}^{m}\frac{1}{u_i(1-u_i)}\right], \quad 0<u_m<\cdots<u_1<1. \tag{26}$$

Proof. Substituting $\Sigma = \lambda I_m$ in (23), and noting that

$$\begin{aligned}{}_0F_0^{(m)}(-\lambda I_m, L^{-1}(I_m-L)^{-1}) &= {}_0F_0(-\lambda L^{-1}(I_m-L)^{-1})\\ &= \mathrm{etr}[-\lambda L^{-1}(I_m-L)^{-1}]\\ &= \exp\left[-\lambda\sum_{i=1}^{m}\frac{1}{u_i(1-u_i)}\right],\end{aligned}$$

we obtain the desired result. □

Theorem 6.3. *If* $V \sim \mathrm{EB2}(m,p,q;\Sigma)$*, then the joint p.d.f. of the eigenvalues* $v_1, v_2, \ldots, v_m$ *of* V *is given by*

$$\frac{\pi^{m^2/2}\,\mathrm{etr}(-2\Sigma)}{\Gamma_m(m/2)B_m(p,q;\Sigma)}\left[\prod_{i<j}^{m}(v_i-v_j)\right]\prod_{i=1}^{m}\left[v_i^{p-(m+1)/2}(1+v_i)^{-(p+q)}\right]$$
$$\times{}_0F_0^{(m)}(-\Sigma, L+L^{-1}), \quad 0<v_m<\cdots<v_1<\infty, \tag{27}$$

where $L = \mathrm{diag}(v_1, v_2, \ldots, v_m)$.

Proof. The p.d.f. of U is given by (10). Applying Theorem 6.1, we obtain the joint p.d.f. of the eigenvalues $v_1, v_2, \ldots, v_m$ of V as

$$\frac{\pi^{m^2/2}\operatorname{etr}(-2\Sigma)}{\Gamma_m(m/2)B_m(p,q;\Sigma)}\left[\prod_{i<j}^{m}(v_i-v_j)\right]\prod_{i=1}^{m}\left[v_i^{p-(m+1)/2}(1+v_i)^{-(p+q)}\right]$$
$$\times\int_{O(m)}\operatorname{etr}[-\Sigma H(L+L^{-1})H'][\mathrm{d}H]. \tag{28}$$

Finally, evaluation of the above integral using (22) yields the desired result. □

Corollary 6.3.1. *If* $V\sim \mathrm{EB2}(m,p,q;\lambda I_m)$*, then the joint p.d.f. of the eigenvalues* $v_1, v_2, \ldots, v_m$ *of* V *is given by*

$$\frac{\pi^{m^2/2}\exp(-2m)}{\Gamma_m(m/2)B_m(p,q;\lambda I_m)}\left[\prod_{i<j}^{m}(v_i-v_j)\right]\prod_{i=1}^{m}\left[v_i^{p-(m+1)/2}(1+v_i)^{-(p+q)}\right]$$
$$\times\exp\left[-\lambda\sum_{i=1}^{m}\left(v_i+\frac{1}{v_i}\right)\right],\quad 0<v_m<\cdots<v_1<\infty. \tag{29}$$

Proof. Substituting $\Sigma=\lambda I_m$ in (27), and observing that

$$\begin{aligned}{}_0F_0^{(m)}(-\lambda I_m, L+L^{-1}) &= {}_0F_0(-\lambda(L+L^{-1}))\\ &= \operatorname{etr}[-\lambda(L+L^{-1})]\\ &= \exp\left[-\lambda\sum_{i=1}^{m}\left(v_i+\frac{1}{v_i}\right)\right],\end{aligned}$$

we get the desired result. □

Theorem 6.4. *If* $W\sim \mathrm{EB3}(m,p,q;\Sigma)$*, then the joint p.d.f. of the eigenvalues* $w_1, w_2, \ldots, w_m$ *of* W *is given by*

$$\frac{2^{mp}\pi^{m^2/2}}{\Gamma_m(m/2)B_m(p,q;\Sigma)}\left[\prod_{i<j}^{m}(w_i-w_j)\right]\prod_{i=1}^{m}\left[\frac{w_i^{p-(m+1)/2}(1-w_i)^{q-(m+1)/2}}{(1+w_i)^{p+q}}\right]$$
$$\times\,{}_0F_0^{(m)}(-\Sigma, 2^{-1}L^{-1}(1-L)^{-1}(1+L)^2),\quad 0<u_m<\cdots<u_1<1, \tag{30}$$

where $L=\operatorname{diag}(w_1,\ldots,w_m)$.

Proof. Similar to the proof of Theorem 6.2. □

Corollary 6.4.1. *If* $W\sim \mathrm{EB3}(m,p,q;\lambda I_m)$*, then the joint p.d.f. of the eigenvalues* $w_1, w_2, \ldots, w_m$ *of* W *is given by*

$$\frac{2^{mp}\pi^{m^2/2}}{\Gamma_m(m/2)B_m(p,q;\lambda I_m)}\left[\prod_{i<j}^{m}(w_i-w_j)\right]\prod_{i=1}^{m}\left[\frac{w_i^{p-(m+1)/2}(1-w_i)^{q-(m+1)/2}}{(1+w_i)^{p+q}}\right]$$
$$\times\exp\left[-\frac{\lambda}{2}\sum_{i=1}^{m}\frac{(1+w_i)^2}{w_i(1-w_i)}\right],\quad 0<w_m<\cdots<w_1<1. \tag{31}$$

Proof. Similar to the proof of Corollary 6.2.1. □

7. SOME INTERESTING MULTIPLE INTEGRALS

Since the density over its support set integrates to one, from (23), (26), (27), (29), (30) and (31), we get several interesting integrals

$$\int \cdots \int_{0<u_m<\cdots<u_1<1} \left[\prod_{i<j}^{m}(u_i-u_j)\right]\prod_{i=1}^{m}\left[u_i^{p-(m+1)/2}(1-u_i)^{q-(m+1)/2}\right] \times {}_0F_0^{(m)}(-\Sigma, L^{-1}(1-L)^{-1})\prod_{i=1}^{m}\mathrm{d}u_i = \frac{\Gamma_m(m/2)B_m(p,q;\Sigma)}{\pi^{m^2/2}},$$

$$\int \cdots \int_{0<u_m<\cdots<u_1<1} \left[\prod_{i<j}^{m}(u_i-u_j)\right]\prod_{i=1}^{m}\left[u_i^{p-(m+1)/2}(1-u_i)^{q-(m+1)/2}\right] \times \exp\left[-\lambda\sum_{i=1}^{m}\frac{1}{u_i(1-u_i)}\right]\prod_{i=1}^{m}\mathrm{d}u_i = \frac{\Gamma_m(m/2)B_m(p,q;\lambda I_m)}{\pi^{m^2/2}},$$

$$\int \cdots \int_{0<v_m<\cdots<v_1<\infty} \left[\prod_{i<j}^{m}(v_i-v_j)\right]\prod_{i=1}^{m}\left[v_i^{p-(m+1)/2}(1+v_i)^{-(p+q)}\right] \times {}_0F_0^{(m)}(-\Sigma, L+L^{-1})\prod_{i=1}^{m}\mathrm{d}v_i = \frac{\Gamma_m(m/2)B_m(p,q;\Sigma)\,\mathrm{etr}(2\Sigma)}{\pi^{m^2/2}},$$

$$\int \cdots \int_{0<v_m<\cdots<v_1<\infty} \left[\prod_{i<j}^{m}(v_i-v_j)\right]\prod_{i=1}^{m}\left[v_i^{p-(m+1)/2}(1+v_i)^{-(p+q)}\right] \times \exp\left[-\lambda\sum_{i=1}^{m}\left(v_i+\frac{1}{v_i}\right)\right]\prod_{i=1}^{m}\mathrm{d}v_i = \frac{\Gamma_m(m/2)B_m(p,q;\lambda I_m)\exp(2m\lambda)}{\pi^{m^2/2}}.$$

$$\int \cdots \int_{0<w_m<\cdots<w_1<1} \left[\prod_{i<j}^{m}(w_i-w_j)\right]\prod_{i=1}^{m}\left[\frac{w_i^{p-(m+1)/2}(1-w_i)^{q-(m+1)/2}}{(1+w_i)^{p+q}}\right] \times {}_0F_0^{(m)}\left(-\Sigma, 2^{-1}L^{-1}(1-L)^{-1}(1+L)^2\right)\prod_{i=1}^{m}\mathrm{d}w_i$$

$$= \frac{\Gamma_m(m/2)B_m(p,q;\Sigma)}{2^{mp}\pi^{m^2/2}},$$

and

$$\int\cdots\int_{0<w_m<\cdots<w_1<1}\left[\prod_{i<j}^{m}(w_i-w_j)\right]\prod_{i=1}^{m}\left[\frac{w_i^{p-(m+1)/2}(1-w_i)^{q-(m+1)/2}}{(1+w_i)^{p+q}}\right]$$
$$\times\exp\left[-\frac{\lambda}{2}\sum_{i=1}^{m}\frac{(1+w_i)^2}{w_i(1-w_i)}\right]\prod_{i=1}^{m}\mathrm{d}w_i$$
$$= \frac{\Gamma_m(m/2)B_m(p,q;\lambda I_m)}{2^{mp}\pi^{m^2/2}}.$$

ACKNOWLEDGEMENTS

This research work was supported by the Sistema Universitario de Investigación, Universidad de Antioquia by the project No. IN10164CE.

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