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An Explicit Solution for Perpetual American Put Options in a Markov-Modulated Jump Diffusion Model

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Abstract: This paper is concerned with the pricing of perpetual American put options when the dynamics of the risky underlying asset are driven by a jump diffusion with Markovian switching. By using the "modified smooth pasting" technique, we derive an explicit optimal stopping rule and the corresponding value function in a closed form. Finally, we present a numerical example to illustrate the application of the exact solution.

Key words: American put option; Regime switching; Optimal stopping time; Markov chain

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1. INTRODUCTION

Though outstandingly successful as a leading-order model for an asset price, the familiar log-Brownian paradigm falls in various ways, such as the fact that implied volatility is not constant. To overcome these shortcomings, many different option valuation models with more realistic price dynamics have been proposed and tested. An important class of stock prices is the Markov-modulated Geometric Brownian Motion (GBM),

$$dX(t) = \mu_{y(t)}X(t)dt + \sigma_{y(t)}X(t)dW(t), \tag{1}$$

where $y(t) \in \{1, \cdots, S\}$ is a finite-state continuous time Markov chain and $W(t)$ is a standard Brownian motion. Such a GBM is also known as the GBM with *Makovian switching*. Some authors call it GBM with regime switching. Here the states of the Markov chain $y(t)$ is called regimes which can be interpreted as the structural changes in economic conditions, the changes in political regimes, the impact of (macro-) economic news and business cycles, etc.. One of the important issues in the study of Makov-modulated GBM is option pricing. There is a large literature in this area, for example [6,8–10], a few to name.

However, empirical studies and large literature show that prices can generate sudden, discontinuous moves. Hence it is more realistic in practice if the 'jumps' in the stock prices are considered. Motivated by [5], [15] and [16], in the present paper, we consider a stock whose price is modeled as

$$dX(t) = \mu_{y(t)}X(t)dt + \sigma_{y(t)}X(t)dW(t) + X(t^-)\int_R z\widetilde{N}(dt, dz), \tag{2}$$

where $y(t) \in \{1, \cdots, S\}$ is a finite-state continuous time Markov chain and $W(t)$ is a standard Brownian motion. Here $y(t)$ and $W(t)$ are defined on $(\Omega, \mathcal{F}, P)$ and are independent. Let $\{\mathcal{F}_t = \sigma\{(W(s), y(s))|s \leq t\}\}$ be an increasing family of sub-σ-algebras of $\mathcal{F}$, and $W(t)$ an $\mathcal{F}_t$ adapted; let $N(dt, dz)$ (corresponding to a random point function $N(t)$) be a stationary $\mathcal{F}_t$ Poisson point process being independent of $W(t)$, and let $\widetilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$ be the compensated Poisson random measure on $[0, \infty) \times R$, where $\nu(.)$ is a deterministic finite characteristic measure on the measurable space $(\mathbb{R} \setminus \{0\}, \mathcal{B}(\mathbb{R} \setminus \{0\}))$. We denote the jump size distribution and the intensity of the compound Poisson process, $F(x)$ and Λ, respectively. To guarantee the existence and uniqueness of the solution to the equation (2), we assume that

$$\int_R |z|^2\nu(dz) < \infty. \tag{3}$$

Moreover, for a given $y(t) = i$, μ_i and σ_i $(i = 1, \cdots, S)$ are constants and known. To the best of our knowledge, there is so far little on America option pricing when stock price is modeled by (2) and our aim here is mainly to close the gap.

As we all know, a perpetual American put option is a derivative that gives its holder the option but not the obligation of exercising a share of stock at his/her choice of time τ $(\tau \geq 0)$, with a payoff $(K - X_\tau)^+ = \max\{0, K - X_\tau\}$. Here, K is the strike price. It is well known that under a risk neutral measure, the value of this option is the expected discounted value of its future cash flow. For more details, readers are referred to Elliott *et al.* [7]. Hence, the optimal stopping problem becomes valuation of the value function

$$V(x, i) = \sup_{0\leq\tau\leq\infty} E[e^{-r\tau}\,(K - X(\tau))^+\,|X(0) = x, y(0) = i], \tag{4}$$

where $r > 0$ is the discounted factor, $X(t)$ is given by (2), and τ is a $\mathcal{F}_t$ stopping time. Elliott *et al.* [7] consider the pricing of a European call option when the risky underlying asset are driven by generalized Markov-modulated jump diffusion model. They use the regime switching generalized Esscher transform to determine an equivalent martingale measure. Moreover, they derive a system of coupled partial-differential-integral equations satisfied by the European option prices.

The primary purpose of this paper is to derive the explicit solution of the value function (4) for $S = 2$. This is an optimal stopping problem with an infinite time horizon and with state space $\{(x, i)|x > 0, i = 1, 2\}$. The methods in this paper depend heavily on negative real roots of the integral differential equation and modified principle of smooth pasting. By using the modified principle smooth pasting, we obtain a closed-form solution. We start in Section 2 to establish the system of partial-integro-differential equation satisfied by the value function, and analyze the form of value function. In Section 3, we establish that the proposed the solution coincides with the value function. At last, a numerical solution is presented.

2. THE DERIVATION OF SOLUTIONS

In this section, we will analyze that the value function (4) when the risky asset $X(t)$ follows the stochastic differential equation (2). Throughout the paper, we assume that the Markov chain $y(t)$ takes two values 1 and 2. Further, we assume that $\sigma_1 \neq \sigma_2$ and that the Markov chain has a generator of the form

$$\begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}, \tag{5}$$

with $\lambda_1, \lambda_2 > 0$.

Recall that when the risky asset follows the Markov-modulated GBM, it is shown that the continuation region depends on the state $y(t)$. It is natural to guess that the continuation region in Markov-modulated jump diffusion model has the similar form. In other words, we expect the existence of two thresholds $x_1, x_2 \leq K$, so that the optimal stopping rule is given as

$$\tau^* = \inf\{t > 0|(X(t), y(t)) \notin D\},$$

where

$$D = \{(x, i)|V^*(x, i) > (K - x)^+\}.$$

The set D is referred to as the *continuation region*. Using τ^*, the corresponding value functions are

$$V^*(x, i) = E[e^{-r\tau^*}(K - X(\tau^*))^+|X(0) = x, y(t) = i]. \tag{6}$$

We consider the case when D can be represented by two threshold levels x_1 and x_2, i.e.,

$$D = \{(x, 1)|x \in (x_1, \infty)\} \cup \{(x, 2)|x \in (x_2, \infty)\}.$$

Notice that x_1 and x_2 may depend on $r,\ K, \mu_i, \sigma_i, \lambda_i$. For any x_1 and x_2, there are only three possibilities $x_1 < x_2$, $x_1 > x_2$, and $x_1 = x_2$. In the rest of this Section, we discuss each of these cases and derive the values of these thresholds x_i as well as the corresponding value functions (denoted as $V(x, i)$ obtained from exercising this type of stopping rule). We will then prove the optimality of these value functions, i.e., $V^*(x, i) = V(x, i)$.

Case 1: $x_1 < x_2 \le K$. At any given time t, if $X(t) \le x_1$, then one should stop immediately and obtain a payoff of $(K - X(t))^+$; this follows from the definition of the continuation region. However, if $X(t) \le x_2$, with $y(t) = 1$, it is not to stop until $X(t) \le x_1$. By Ito's formula, we can follow the value functions $V(x, 1)$ and $V(x, 2)$ satisfy the following partial-differential-integral equations. For $x \in [x_2, +\infty)$

$$\begin{cases} \mu_1 x V'_x(x,1) + \frac{1}{2}\sigma_1^2 x^2 V''_{xx}(x,1) + \int_R [V(x+xz,1) - V(x,1)]\nu(dz) \\ -(\lambda_1 + r_1)V(x,1) + \lambda_1 V(x,2) = 0, \\ \mu_2 x V'_x(x,2) + \frac{1}{2}\sigma_2^2 x^2 V''_{xx}(x,2) + \int_R [V(x+xz,2) - V(x,2)]\nu(dz) \\ -(\lambda_2 + r)V(x,2) + \lambda_2 V(x,1) = 0; \end{cases} \tag{7}$$

for $x \in [x_1, x_2]$, we have

$$\begin{cases} \mu_1 x V'_x(x,1) + \frac{1}{2}\sigma_1^2 x^2 V''_{xx}(x,1) + \int_R [V(x+xz,1) - V(x,1)]\nu(dz) \\ -(\lambda_1 + r_1)V(x,1) + \lambda_1 V(x,2) = 0, \\ V(x,2) = K - x; \end{cases} \tag{8}$$

and for $x \in [0, x_1]$

$$V(x,1) = V(x,2) = K - x. \tag{9}$$

Now, let us solve the system (7). Inspired by Guo and Zhang [10], we introduce the following characteristic function

$$g_1(\beta)g_2(\beta) = \lambda_1\lambda_2, \tag{10}$$

where

$$g_1(\beta) = \lambda_1 + r - \mu_1\beta - \frac{1}{2}\sigma_1^2\beta(\beta-1) - \int_R \left[(1+z)^\beta - 1\right]\nu(dz),$$

$$g_2(\beta) = \lambda_2 + r - \mu_2\beta - \frac{1}{2}\sigma_2^2\beta(\beta-1) - \int_R \left[(1+z)^\beta - 1\right]\nu(dz).$$

We will see that in this case the solutions to the value functions are closely related to the roots of

$$g_1(\beta) = 0. \tag{11}$$

Proposition 2.1. *The equation (11) has at least two real roots, of which has a unique negative root, say γ_1.*

Proof. The continuity of function $g_1(\beta)$ on $(-\infty, +\infty)$, $\lim_{\beta\to-\infty} g_1(\beta) = \lim_{\beta\to+\infty} g_1(\beta) = -\infty$, and the fact that $g(0) = \lambda_1 + r$, imply that the existence of two roots in equation (11). We conclude that equation (11) has at least two roots.

It remains to prove the uniqueness of the negative real root. Let $p_1(\beta) = \lambda_1 + r - \mu_1\beta - \frac{1}{2}\sigma_1^2\beta(\beta-1)$. On the half circle in the complex plane given by $z = r$ (for $r > 0$ fixed) and $\mathbb{R}(z) \le 0$) we have $|p(s)| > 2|\int_R (1+z)\nu(dz)| \ge |\int_R [(1+z)^\beta - 1]\nu(dz)|$. By Rouché Theorem, equation $g_1(\beta) = 0$ and equation $p_1(\beta) = 0$ have the same number of zeros on the interior of half circle. Since the later has a unique negative real root $s_1 = \dfrac{\frac{1}{2}\sigma_1^2 - \mu_1 - \sqrt{(\frac{1}{2}\sigma_1^2 - \mu_1)^2 + 2\sigma_1^2(\lambda_1 + r)}}{\sigma_1^2}$, this completes the proof. ∎

For notational convenience, denote γ_2 the least positive root of equation (11).

Proposition 2.2. *If $\sigma_1,\ \sigma_2,\ \lambda_1,\ \lambda_2$ are positive constants, then the equation*

$$g_1(\beta)g_2(\beta) = \lambda_1\lambda_2 \tag{12}$$

has at least four distinct roots, of which has exactly two negative roots, say $\beta_1,\ \beta_2$.

Proof. We note that $\lim_{\beta\to+\infty} g_1(\beta) = -\infty$, $\lim_{\beta\to-\infty} g_1(\beta) = -\infty$, and $g_1(0) = \lambda_1 + r$. The continuity of $g_1(\theta)$ implies that there are two roots which satisfy equation $g_1(\theta) = 0$.

Let

$$\begin{aligned} f(\beta) = &\left[\lambda_1 + r - \mu_1\beta - \tfrac{1}{2}\sigma_1^2\beta(\beta-1) - \textstyle\int_R \left[(1+z)^\beta - 1\right]\nu(dz)\right] \\ &\times \left[\lambda_2 + r - \mu_2\beta - \tfrac{1}{2}\sigma_2^2\beta(\beta-1) - \textstyle\int_R \left[(1+z)^\beta - 1\right]\nu(dz)\right] - \lambda_1\lambda_2. \end{aligned}$$

Let θ_1, θ_2 be the roots of the corresponding quadratic equation

$$g_1(\beta) = \lambda_1 + r - \mu_1\beta - \frac{1}{2}\sigma_1^2\beta(\beta-1) - \int_R [(1+z)^\beta - 1]\nu(dz) = 0. \tag{13}$$

Clearly, the continuous function $f(\beta)$ satisfies $f(0) = r^2 + (\lambda_1+\lambda_2)r > 0$, $f(-\infty) > 0$, $f(+\infty) > 0$ and $f(\theta_i) = -\lambda_1\lambda_2 < 0$ for $i = 1, 2$. Since $\theta_1\theta_2 < 0$, it follows that the equation $f(\beta) = 0$ has at least four real roots. Next, we will show the number of negative real roots of equation (13) is two. Similarly the proof above, put $p_i(\beta) = \lambda_i + r - \mu_i\beta - \frac{1}{2}\sigma_i^2\beta(\beta-1),\ i = 1, 2$. Considering the half circle in the complex $\Gamma = \{z||z| = r, Re(z) < 0\}\bigcup\{z||z| \le r, \mathbb{R}(z) = 0\}$, when r is sufficiently large, it is easy to check that on the closed contour Γ

$$\begin{aligned} &|p_1(\beta)p_2(\beta)| > \\ &\left|-\int_R \left[(1+z)^\beta - 1\right]\nu(dz)\,(p_1(\beta)+p_2(\beta)) + \left\{\int_R \left[(1+z)^\beta - 1\right]\nu(dz)\right\}^2 - \lambda_1\lambda_2\right| \end{aligned}$$

By the Rouché's Theorem, equation

$$\begin{aligned} p_1(\beta)p_2(\beta) - &\int_R \left[(1+z)^\beta - 1\right]\nu(dz)\,(p_1(\beta)+p_2(\beta)) \\ &+ \left\{\int_R \left[(1+z)^\beta - 1\right]\nu(dz)\right\}^2 - \lambda_1\lambda_2 = 0 \end{aligned}$$

and equation $p_1(\beta)p_2(\beta) = 0$ have the same number of zeros inside Γ. This completes the proof. □

For simplicity, we number the positive roots by $\beta_3, \beta_4, \cdots, \beta_n, n \ge 3$.

Proposition 2.3. *The value function $V(x, i)$ defined by (6) satisfy the principe of smooth pasting.*

Proof. For given $y(t) = i$, the risky asset process $X(t)$ is unbounded variation. By Proposition 7 in [1], the point 0 is regular for $(-\infty, 0)$. Again by using Theorem 6 in [1], the result is followed. □

To solve the system (7), we try the value functions of the form

$$V(x,1) = \sum_{i=1}^{n} A_i x^{\beta_i}, \tag{14}$$

$$V(x,2) = \sum_{i=1}^{n} B_i x^{\beta_i}, \tag{15}$$

with $B_i = l_i A_i$ and $l_i = \dfrac{g_1(\beta_i)}{\lambda_1} A_i = \dfrac{g_2(\beta_i)}{\lambda_2} A_i$.

Note that when $x \to \infty$, $V(x,1)$ and $V(x,2)$ are bounded. Thus, the positive powers of x should be eliminated so that

$$V(x,1) = A_1 x^{\beta_1} + A_2 x^{\beta_2}, \tag{16}$$

$$V(x,2) = B_1 x^{\beta_1} + B_2 x^{\beta_2}. \tag{17}$$

Next, we get down to solve (8). The first equation is an inhomogenous integral-differential equation. By Proposition 2.1, we can consider the solution of the following form

$$V(x,1) = C_1 x^{\gamma_1} + C_2 x^{\gamma_2} + \psi(x),$$

where $\psi(x)$ is a special solution and γ_1, γ_2 are given by Proposition 2.1.

In particular, when $\mu_1 - \lambda_1 - r \neq 0$, one can choose

$$\psi(x) = \frac{\lambda_1 + \int_R z\nu(dz)}{\mu_1 - \lambda_1 - r} x + \frac{\lambda_1 K}{\lambda_1 + r}. \tag{18}$$

Now, we would like to determine $A_1, A_2, C_1, C_2, x_1, x_2$. To this end, appropriated boundary conditions are needed. Applying the smooth pasting at x_2, condition $V(x_2+,2) = V(x_2-,2)$ and $V'(x_2+,2) = V'(x_2-,2)$ imply

$$\begin{cases} B_1 x_2^{\beta_1} + B_2 x_2^{\beta_2} = K - x_2, \\ \beta_1 B_1 x_2^{\beta_1 - 1} + \beta_2 B_2 x_2^{\beta_2 - 1} = -1. \end{cases} \tag{19}$$

Similarly, the smoothness of $V(x,1)$ at x_1 and x_2 yields

$$\begin{cases} A_1 x_2^{\beta_1} + A_2 x_2^{\beta_2} = C_1 x_2^{\gamma_1} + C_2 x_2^{\gamma_2} + \psi(x_2), \\ \beta_1 A_1 x_2^{\beta_1 - 1} + \beta_2 A_2 x_2^{\beta_2 - 1} = C_1 \gamma_1 x_2^{\gamma_1 - 1} + C_2 \gamma_2 x_2^{\gamma_2 - 1} + \psi'(x_2), \end{cases} \tag{20}$$

and

$$\begin{cases} C_1 x_1^{\gamma_1} + C_2 x_1^{\gamma_2} + \psi(x_1) = K - x_1, \\ \gamma_1 C_1 x_1^{\gamma_1} + \gamma_2 C_2 x_1^{\gamma_2} + x_1 \psi'(x_1) = -x_1. \end{cases} \tag{21}$$

Combing the above three equations and following some algebraic manipulation, we obtain an algebraic equation for x_1 and x_2

$$\begin{pmatrix} x_1^{-\gamma_1} & 0 \\ 0 & x_1^{-\gamma_2} \end{pmatrix} F_1(x_1) = \begin{pmatrix} x_2^{-\gamma_1} & 0 \\ 0 & x_2^{-\gamma_2} \end{pmatrix} F_2(x_2), \tag{22}$$

where

$$F(x_1) = \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \begin{pmatrix} K - x_1 - \psi(x_1) \\ -x_1 - x_1\psi'(x_1) \end{pmatrix} \tag{23}$$

and

$$F_2(x_2) = \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \left[\begin{pmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} l_1 & l_2 \\ \beta_1 l_1 & \beta_2 l_2 \end{pmatrix}^{-1} \begin{pmatrix} K - x_2 \\ -x_2 \end{pmatrix} - \begin{pmatrix} \psi(x_2) \\ x_2 \psi'(x_2) \end{pmatrix} \right]. \tag{24}$$

In particular, if $\mu_1 - \lambda_1 - r \neq 0$, where $\psi(x_1)$ is in the form of (18), then

$$F_1(x_1) = a_1 + a_2 x_1$$

and

$$F_2(x_2) = b_1 + b_2 x_2.$$

Here

$$a_1 = \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \begin{pmatrix} \frac{rK}{r+\lambda_1} \\ 0 \end{pmatrix},$$

$$a_2 = - \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \begin{pmatrix} 1 + \frac{\lambda_1 + \int_R z\nu(dz)}{\mu_1 - \lambda_1 - r} \\ 1 + \frac{\lambda_1 + \int_R z\nu(dz)}{\mu_1 - \lambda_1 - r} \end{pmatrix},$$

$$b_1 = \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \left[\begin{pmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} l_1 & l_2 \\ l_1\beta_1 & l_2\beta_2 \end{pmatrix}^{-1} \begin{pmatrix} K \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{\lambda_1 K}{\lambda_1 + r} \\ 0 \end{pmatrix} \right],$$

$$b_2 = \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \left[\begin{pmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} l_1 & l_2 \\ l_1\beta_1 & l_2\beta_2 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ -1 \end{pmatrix} - \begin{pmatrix} \frac{\lambda_1 + \int_R z\nu(dz)}{\mu_1 - \lambda_1 - r} \\ \frac{\lambda_1 + \int_R z\nu(dz)}{\mu_1 - \lambda_1 - r} \end{pmatrix} \right]$$

The coefficients are given by

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} l_1 x_2^{\beta_1} & l_2 x_2^{\beta_2} \\ l_1 \beta_1 x_2^{\beta_1} & l_1 \beta_2 x_2^{\beta_2} \end{pmatrix}^{-1} \begin{pmatrix} K - x_2 \\ -x_2 \end{pmatrix},$$

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} l_1 A_1 \\ l_2 A_2 \end{pmatrix},$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} x_1^{-\gamma_1} & 0 \\ 0 & x_1^{-\gamma_2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \begin{pmatrix} K - x_1 - \psi(x_1) \\ -x_1 - x_1\psi'(x_1) \end{pmatrix}.$$

With these coefficients, the value functions become

$$V(x,1) = \begin{cases} A_1 x^{\beta_1} + A_2 x^{\beta_2} & \text{if } x > x_2, \\ C_1 x^{\gamma_1} + C_2 x^{\gamma_2} + \psi(x) & \text{if } x_1 < x \leq x_2, \\ K - x & \text{if } x \leq x_1, \end{cases} \tag{25}$$

$$V(x,2) = \begin{cases} B_1 x^{\beta_1} + B_2 x^{\beta_2} & \text{if } x > x_2, \\ K - x & \text{if } x \leq x_2. \end{cases} \tag{26}$$

Case 2: $x_2 < x_1 \leq K$. The derivation of this case is analogous to that of $x_1 < x_2$, and we only summarize the results below. By Proposition 2.1, we let $\widetilde{\gamma}_1$,

$\widetilde{\gamma}_2$ (similar to definition of γ_1 and γ_2. In addition, denote γ_2 the least positive root) be the two real roots of

$$\mu_2\gamma + \frac{1}{2}\sigma_2^2\gamma^2 + \int_R [(1+z)^\gamma - 1]\nu(dz) - (\lambda_2 + r) = 0,$$

and $\widetilde{\psi}(x)$ be a particular solution to

$$V_x'(x,2) + \frac{1}{2}\sigma^2 x^2 V_{xx}''(x,2) + \int_R [V(x+xz,2) - V(x,2)]\nu(dz) - (\lambda_2 + r)V(x,2) + \lambda_2(K - x) = 0.$$

In particular, when $\mu_2 - \lambda_2 - r \neq 0$, one can choose

$$\widetilde{\psi}(x) = \frac{\lambda_2 + \int_R z\nu(dz)}{\mu_2 - \lambda_2 - r}x + \frac{\lambda_2 K}{\lambda_2 + r}. \tag{27}$$

In addition, x_1, x_2 satisfy

$$\begin{pmatrix} x_1^{-\widetilde{\gamma_1}} & 0 \\ 0 & x_1^{-\widetilde{\gamma_2}} \end{pmatrix} \widetilde{F}_1(x_1) = \begin{pmatrix} x_2^{-\widetilde{\gamma_1}} & 0 \\ 0 & x_2^{-\widetilde{\gamma_1}} \end{pmatrix} \widetilde{F}_2(x_2), \tag{28}$$

where

$$\widetilde{F}_1(x_1) = \begin{pmatrix} 1 & 1 \\ \widetilde{\gamma}_1 & \widetilde{\gamma}_2 \end{pmatrix}^{-1} \left[\begin{pmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} \tilde{l}_1 & \tilde{l}_2 \\ \beta_1\tilde{l}_1 & \beta_2\tilde{l}_2 \end{pmatrix}^{-1} \begin{pmatrix} K - x_1 \\ -x_1 \end{pmatrix} - \begin{pmatrix} \widetilde{\psi}(x_1) \\ x_1\widetilde{\psi}'(x_1) \end{pmatrix} \right], \tag{29}$$

and

$$\widetilde{F}_2(x_2) = \begin{pmatrix} 1 & 1 \\ \widetilde{\gamma}_1 & \widetilde{\gamma}_2 \end{pmatrix}^{-1} \begin{pmatrix} K - x_2 - \widetilde{\psi}(x_2) \\ -x_2 - x_2\widetilde{\psi}'(x_2) \end{pmatrix}, \tag{30}$$

where $\tilde{l}_i = \dfrac{1}{l_i}$.

In particular, if $\mu_2 - \lambda_2 - r \neq 0$, where $\psi(x_1)$ is in the form of (27), then

$$\widetilde{F}_1(x_1) = \widetilde{a}_1 + \widetilde{a}_2 x_1$$

and

$$\widetilde{F}_2(x_2) = \widetilde{b}_1 + \widetilde{b}_2 x_2.$$

Here

$$\widetilde{a}_1 = \begin{pmatrix} 1 & 1 \\ \widetilde{\gamma}_1 & \widetilde{\gamma}_2 \end{pmatrix}^{-1} \left[\begin{pmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} \widetilde{l}_1 & \widetilde{l}_2 \\ \widetilde{l}_1\beta_1 & \widetilde{l}_2\beta_2 \end{pmatrix}^{-1} \begin{pmatrix} -K \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{\lambda_2 K}{\lambda_2 + r} \\ 0 \end{pmatrix} \right],$$

$$\widetilde{a}_2 = \begin{pmatrix} 1 & 1 \\ \widetilde{\gamma}_1 & \widetilde{\gamma}_2 \end{pmatrix}^{-1} \left[\begin{pmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} \widetilde{l}_1 & \widetilde{l}_2 \\ \widetilde{l}_1\beta_1 & \widetilde{l}_2\beta_2 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ -1 \end{pmatrix} - \begin{pmatrix} \frac{\lambda_2 + \int_R z\nu(dz)}{\mu_2 - \lambda_2 - r} \\ \frac{\lambda_2 + \int_R z\nu(dz)}{\mu_2 - \lambda_2 - r} \end{pmatrix} \right],$$

$$\widetilde{b}_1 = \begin{pmatrix} 1 & 1 \\ \widetilde{\gamma}_1 & \widetilde{\gamma}_2 \end{pmatrix}^{-1} \begin{pmatrix} \frac{rK}{r+\lambda_2} \\ 0 \end{pmatrix},$$

$$\widetilde{b}_2 = -\begin{pmatrix} 1 & 1 \\ \widetilde{\gamma}_1 & \widetilde{\gamma}_2 \end{pmatrix}^{-1} \begin{pmatrix} 1 + \frac{\lambda_2 + \int_R z\nu(dz)}{\mu_2 - \lambda_2 - r} \\ 1 + \frac{\lambda_2 + \int_R z\nu(dz)}{\mu_2 - \lambda_2 - r} \end{pmatrix}.$$

The coefficients are given by

$$\begin{pmatrix} \widetilde{A}_1 \\ \widetilde{A}_2 \end{pmatrix} = \begin{pmatrix} l_1 x_1^{\beta_1} & l_2 x_1^{\beta_2} \\ l_1\beta_1 x_1^{\beta_1} & l_2\beta_2 x_1^{\beta_2} \end{pmatrix}^{-1} \begin{pmatrix} K - x_2 \\ -x_2 \end{pmatrix}$$

$$\begin{pmatrix} \widetilde{B}_1 \\ \widetilde{B}_2 \end{pmatrix} = \begin{pmatrix} l_1\widetilde{A}_1 \\ l_2\widetilde{A}_2 \end{pmatrix}.$$

$$\begin{pmatrix} \widetilde{C}_1 \\ \widetilde{C}_2 \end{pmatrix} = \begin{pmatrix} x_1^{-\widetilde{\gamma}_1} & 0 \\ 0 & x_1^{-\widetilde{\gamma}_2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \begin{pmatrix} K - x_1 - \widetilde{\psi}(x_1) \\ -x_1 - x_1\widetilde{\psi}'(x_1) \end{pmatrix}.$$

With these coefficients, the value functions become

$$V(x,1) = \begin{cases} \widetilde{A}_1 x^{\beta_1} + \widetilde{A}_2 x^{\beta_2} & \text{if } x > x_1, \\ K - x & \text{if } x \le x_1, \end{cases} \tag{31}$$

$$V(x,2) = \begin{cases} \widetilde{B}_1 x^{\beta_1} + \widetilde{B}_2 x^{\beta_2} & \text{if } x > x_1, \\ \widetilde{C}_1 x^{\widetilde{\gamma}_1} + \widetilde{C}_2 x^{\widetilde{\gamma}_2} + \widetilde{\psi}(x) & \text{if } x_2 < x \le x_1, \\ K - x & \text{if } x \le x_2. \end{cases} \tag{32}$$

Case 3: $x_1 = x_2 \le K$. In this case, the continuation region D becomes

$$D = \{(x,i) : x > x^*\}.$$

Therefore, for $x \ge x^*$, we get

$$V(x,1) = A_1 x^{\beta_1} + A_2 x^{\beta_2},$$
$$V(x,2) = B_1 x^{\beta_1} + B_2 x^{\beta_2}$$

and $V(x,1) = V(x,2) = K - x$ for $x \in [0, x^*]$. By smooth pasting at x^*, we can follow $A_1 = B_1$, $A_2 = B_2$, and therefore, $V(x,1) = V(x,2)$. Put $V(x) = V(x,1)$, then for $x > x^*$, the system (7) can be written as

$$\mu_i x V_x'(x) + \frac{1}{2}\sigma_i^2 x^2 V_{xx}''(x) + \int_R [V(x+xz) - V(x)]\,\nu(dz) - rV(x,z) = 0$$

for both $i = 1, 2$. It is easy to check

$$V(x) = V(x,1) = V(x,2) = \begin{cases} \frac{(K-x^*)x^\beta}{(x^*)^\beta} & \text{if } x > x^*, \\ K - x & \text{if } x \le x^*, \end{cases}$$

where $x^* = \frac{K\beta}{\beta-1}$ and β is the unique negative real root of

$$\mu_i\beta + \frac{1}{2}\sigma_i^2\beta(\beta-1) + \int_R [(1+z)^\beta - 1]\nu(dz) - r = 0.$$

3. A VERIFICATION THEOREM

In this section, we will show that the value functions derived in the previous Section is optimal. For any $v(x,i)$ such that $v(.,i) \in C^2(R)$, define

$$\begin{aligned} \mathcal{A}v(x,i) &= x\mu_i \frac{\partial v(x,i)}{\partial x} + \frac{1}{2}x^2\sigma_i^2 \frac{\partial^2 v(x,i)}{\partial x^2} + \int_R \{v(x+xz,i) - v(x,i)\}\,\nu(dz) \\ &\quad + \lambda_i(v(x,3-i) - v(x,i)) - rv(x,i). \end{aligned}$$

Theorem 3.1. *Suppose that (22) (resp. (28)) has a solution (x_1^*, x_2^*) such that $0 < x_1^* \le K$ and $0 < x_2^* \le K$. Define*

$$D = \{(x,i) | v(x,i) > (K-x)^+\},$$
$$S = \{(x,i) | x \ge 0, i = 1,2\},$$

If we can find a function $v : S \to \mathbb{R}$ such that

$$v(x,i) \in C^2(S \setminus \partial D) \cap C^1(S) \tag{33}$$
$$v(x,i) > (K-x)^+ \text{ on } S, \tag{34}$$
$$\mathcal{A}v(x,i) \le 0 \text{ on } S \setminus \partial D, \tag{35}$$
$$\mathcal{A}v(x,i) \le 0 \text{ on } D, \tag{36}$$

Moreover, assume

$$\tau_D := \inf\{t \ge 0 | (X(t), y(t)) \notin D\} < \infty \text{ a.s. for } (x,i) \in D. \tag{37}$$

Then

$$v(x,i) = V(x,i)$$

and

$$\tau^* = \tau_D$$

is an optimal stopping time.

Proof. It is easy to see that $v(\infty, i) = 0,\ i = 1,2$, and

$$D = \{(x,1) | x \in (x_1^*, \infty)\} \cup \{(x,2) | x \in (x_2^*, \infty)\}.$$

Let τ be any stopping time. By a smooth approximation approach for variational inequalities in Øksendal and Sulem [16] we can assume that $v(.,i) \in C^2$. Then by the Dynkin formula applied to $\tau_k := \min(\tau, k); k = 1, 2, \cdots$, we have,

$$E^{(x,i)}[v(X(\tau_k))] = v(x,i) + E^{(x,i)}\left[\int_0^{\tau_k} \mathcal{A}v(X(\tau_s))ds\right].$$

Hence by (35) and the Fatou lemma

$$\begin{aligned} v(x,i) &\ge \liminf_{k\to\infty} E^{(x,i)}[v(X(\tau_k))] \\ &\ge E^{(x,i)}[(K - X(\tau))^+)]. \end{aligned}$$

Hence

$$v(x,i) \ge E^{(x,i)}[e^{-r\tau^*}(K - X(\tau^*)^+]. \tag{38}$$

To show the optimality of τ^*, if we apply the above argument to $\tau = \tau_D$, then by (36) and (37) we get equality in (38), such that

$$v(x,i) \le E[e^{-r\tau^*}(K - X(\tau^*))^+].$$

Combining this with (38), we have $v(x,i) = E[e^{-r\tau^*}(K - X(\tau^*))^+]$. This completes the proof.

From the discussion of the previous Sections, it is easy to check that the value functions $V(x,i)$ satisfy all the conditions of Theorem 3.1. □

4. A NUMERICAL EXAMPLE

In this Section, we present a numerical example. We assume the jump size of stock process follows the Pareto distribution $F(z) = 1 - (1+z)^{-2}$, $\Lambda = 1$, that is $\nu(dz) = \dfrac{2}{(1+z)^3}dz$. Then the equation (13) can be rewritten as

$$\begin{aligned} f(\beta) &= \left[(\lambda_1 + r) - (\mu_1 + \frac{1}{2}\sigma_1^2)\beta - \frac{1}{2}\sigma_1^2\beta^2 + 1 + \frac{2}{\beta - 2}\right] \\ &\times \left[(\lambda_2 + r) - (\mu_2 + \frac{1}{2}\sigma_2^2)\beta - \frac{1}{2}\sigma_2^2\beta^2 + 1 + \frac{2}{\beta - 2}\right] - \lambda_1\lambda_2. \end{aligned} \tag{39}$$

Let

$$r = 0.25, \sigma_1 = 15, \sigma_2 = 16, \lambda_1 = 50, \lambda_2 = 150, \mu_1 = 30, \mu_2 = 50, K = 5.$$

After some calculations, we find the following thresholds: $x_1^* = 0.0509, x_2^* = 0.0672$ and the corresponding value functions:

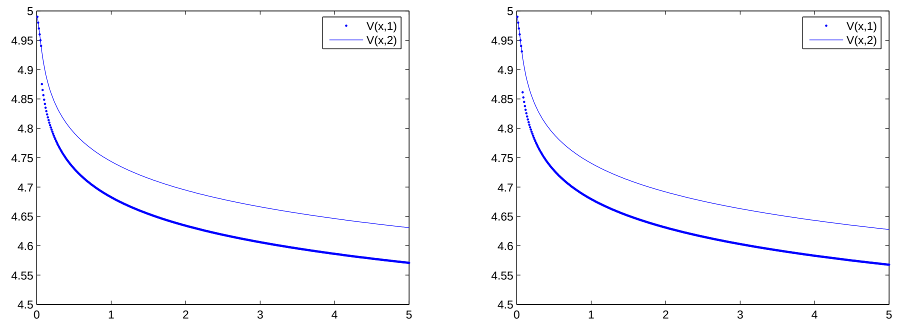
$$V(x,1) = \begin{cases} 0.00017x^{-0.99371} + 4.68250x^{-0.01499} & \\ \qquad\qquad\qquad\qquad\qquad \text{if } x > 0.06725, \\ 0.00762x^{-0.39855} + 2.20948x^{1.1388} - 2.49383x + 4.97512 & \\ \qquad\qquad\qquad\qquad\qquad \text{if } 0.05098 < x \le 0.067, \\ 5 - x \qquad\qquad\qquad\qquad \text{if } x < 0.0509, \end{cases}$$

$$V(x,2) = \begin{cases} -0.0049x^{-0.99371} + 4.7400x^{-0.01499} & \text{if } x > 0.067, \\ 5 - x & \text{if } x \le 0.067. \end{cases}$$

The numerical results are plotted in Figure 1 (a). In contrast, the value functions with $\lambda_1 = 55$, other parameters as in the example above. It can be seen from Figure 1, $V(x,2) \ge V(x,1)$.

5. CONCLUSIONS

In this paper, we derive an explicit formula for perpetual American put options on an asset whose price is modeled by jump-diffusions with Markovian switching. This model can be viewed as generalization of Markov-modulated GBM and jump-diffusions without regime switching. Our method provides a sufficient condition to find the explicit solution in this model. This techniques we use provide a way to discussion a high order integro-differential equation that often arises in the asset pricing theory. Although the value functions presented are still hard to find the exact solutions, this formula lays a theoretic base to numerical approximation.



(a) $\lambda_1 = 50, \lambda_2 = 150$

(b) $\lambda_1 = 55, \lambda_2 = 150$

Figure 1
Value Functions

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