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One Conclusion on the Essential Singularity of Analytic Function

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Abstract
In the article, the isolated singularity, removable singularity, zero, pole, essential singularity and other concepts and properties were used; Two lemmas on the isolated singularity were proved first, and to use these lemmas, one property of essential singularity was proved then.

Key words
Analytic function; Isolated singularity; Essential singularity; Laurent expansion

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1. INTRODUCTION

The deep research shows that theory of complex function has extensive application in physics, engineering, and other subject[1−6].

As the important research content, in the article, the isolated singularity, removable singularity, zero, pole, essential singularity and other concepts and properties[7−8] were used; Two lemmas on the isolated singularity were proved first, and to use these lemmas, one property of essential singularity was proved then.

2. THE PROOF OF LEMMAS; PREPARATION

Lemma 1 Let f be analytic on a region A and have an isolated singularity[1−6] at z_0.

(i) z_0 is a removable singularity[9−10] iff any one of the following conditions holds: (1) f is bounded in a deleted neighborhood[11−14] of z_0; (2) $\lim\limits_{z\to z_0} it\ f(z)$ exists; or (3) $\lim\limits_{z\to z_0} it(z-z_0)f(z) = 0$. (Note that it is not immediately evident that these three conditions are equivalent but the assertion is that they are and that each is equivalent to the condition that f has a removable singularity.

(ii)z_0 is a simple pole iff $\lim\limits_{z\to z_0} it(z-z_0)f(z)$ exists and is unequal to zero. This limit equals the residue of f at z_0.

(iii) z_0 is a pole of order $\leq k$(or possibly a removable singulari0) iff any one of the following conditions holds:(1) There is a constant M>0 and an integer $k \geq 1$ such that $f(z) \leq \frac{M}{|z-z_0|^k}$. in a deleted neighborhood of z_0; (2)$\lim\limits_{z\to z_0} it(z-z_0)^{k+1} f(z) = 0$; or (3)$\lim\limits_{z\to z_0} it(z-z_0)^{k+1} f(z)$ exists.

(iv) z_0 is a pole of order $k \geq 1$ iff there is an analytic function[15] φ defined on a neighborhood U of z_0 such that $U\backslash\{z_0\} \subset A$ such that $\varphi(z_0) \neq 0$, and such that

$$f(z) = \frac{\varphi(z)}{(z-z_0)^k} \text{ for } z \in U,\ z \neq z_0.$$

Proof. (i)If z_0 is a removable singularity, then in a deleted neighborhood of z_0 we have $f(z) = \sum\limits_{n=0}^{\infty} a_n(z-z_0)^n$ Since this series represents an analytic function in an undeleted neighborhood of z_0, obviously conditions (1), (2), and (3) hold. Conditions (1) and (2) each obviously implies condition (3), so it remains to be shown that (3) implies that z_0 is a removable singularity for f. We must prove that each b_k in the Laurent expansion of f around z_0 is 0. Now$b_k = \int_{\gamma_r} f(\zeta)(\zeta-z_0)^{k-1} d\zeta$, where γ_r is a circle whose interior (except for z_0) lies in A. Let ε¿ 0 be given. By condition (3) can choose r>0 with r < 1 such that γ_r we have $|f(\zeta)| < \frac{\varepsilon}{|\zeta-z_0|} = \frac{\varepsilon}{r}$. Then

$$|b_k| \leq \frac{1}{2\pi}\int_{\gamma_r} |f(\zeta)||\zeta-z_0|^{k-1}|d\zeta| \leq \frac{1}{2\pi}\frac{\varepsilon}{r} r^{k-1}\int_{\gamma_r}|d\zeta|$$
$$= \frac{1}{2\pi}\frac{\varepsilon}{r} r^{k-1} 2\pi r = \varepsilon r^{k-1} \leq \varepsilon$$

Thus $|b_k| \leq \varepsilon$, Since ε was arbitrary b_k=0. We shall use (iii) to prove (ii). so (iii) is proved next.

(iii) This proof follows immediately by applying (i) to the function $(z-z_0)^k f(z)$, which is analytic on A. (One can easily obtained the details of the proof) (ii) If z_0 is a simple pole, then in a deleted neighborhood of z_0 we have

$$f(z) = \frac{b_1}{z-z_0} + \sum_{n=0}^{\infty} a_n(z-z_0)^n = \frac{b_1}{z-z_0} + h(z)$$

where h is analytic at z_0 and where $b_1 \neq 0$ by the Laurent expansion. Hence

$$\lim_{z\to z_0} it(z-z_0)f(z) = \lim_{z\to z_0} it(b_1 + (z-z_0)h(z)) = b_1.$$

On the other hand, suppose that $\lim\limits_{z\to z_0} it(z-z_0)f(z)$ exists and is unequal to zero. Thus $\lim\limits_{z\to z_0} it(z-z_0)^2 f(z) = 0$. By the result obtained in (iii), this says that

$$f(z) = \frac{b_1}{z-z_0} + \sum_{n=0}^{\infty} a_n(z-z_0)^n = \frac{b_1}{z-z_0} + h(z)$$

for some constant b_1, and analytic function h. where b_1 may or may not be zero. But then $(z-z_0)f(z) = b_1 + (z-z_0)h(z)$, so $\lim\limits_{z\to z_0} it(z-z_0)f(z) = b_1$. Thus, in fact, $b_1 \neq 0$, and therefore f has a simple pole at z_0.

(iv) z_0 is a pole of order $k \geq 1$ iff

$$f(z) = \frac{b_k}{(z-z_0)^k} + \frac{b_{k-1}}{(z-z_0)^{k-1}} + \cdots + \frac{b_1}{(z-z_0)} + \sum_{n=0}^{\infty} a_n(z-z_0)^n$$
$$= \frac{1}{(z-z_0)^k}\left\{b_k + b_{k-1}(z-z_0) + \cdots + b_1(z-z_0)^{k-1} + \sum_{n=0}^{\infty} a_n(z-z_0)^{n+k}\right\}, \quad (b_k \neq 0)$$

(where $b_k \neq 0$). This expansion is valid in a deleted neighborhood of z_0. $\varphi(z) = b_k + b_{k-1}(z-z_0) + \cdots + b_1(z-z_0)^{k-1} + \sum\limits_{n=0}^{\infty} a_n(z-z_0)^{n+k}$. Then $\varphi(z)$ is analytic in the corresponding undeleted neighborhood (since it is a convergent power series) and $\varphi(z_0) = b_k \neq 0$. Conversely, given such a φ, we can retrace these steps to show that

z_0 is a pole of order $k \geq 1$.

Definition 1 Let f be analytic on a region A and let $z_0 \in A$. We say that f has a zero of order k at z_0 iff $f(z_0) = 0, \ldots, f^{(k-1)}(z_0) = 0, f^{(k)}(z_0) \neq 0$.

From the Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(k)}(z_0)}{n!}(z - z_0)^n$$

we see that f has a zero of order k iff, in a neighborhood of z_0, we can write $f(z) = (z - z_0)^k g(z)$ where $g(z)$ is analytic at z_0 and $g(z_0) = \frac{f^{(k)}(z_0)}{k!} \neq 0$. Thus from **Lemma 1** (iv). let $\varphi(z) = g(z)^{-1}$ we get the following.

Lemma 2 If f is analytic in a neighborhood of z_0, then f has a zero of order k at z_0 iff $\frac{1}{f(z)}$ has a pole of order k at z_0. If h is analytic and $h(z_0) \neq 0$, then $\frac{h(z)}{f(z)}$ also has a pole of order k.

Obviously, if z_0 is a zero of f and f is not identically equal to zero in a neighborhood of z_0, then z_0 has some finite order k. (Otherwise the Taylor series would be identically zero.)

Definition 1 In practical problems we usually are dealing with a pole. It is not hard to show that if $f(z)$ has a pole (of finite order k) at z_0, then $|f(z)| \to \infty$ as $z \to z_0$. However, in case of an essential singularity, $|f|$ will not, in general, approach ∞, as $z \to z_0$. In fact, we have the following result.

3. FINAL THEOREM AND ITS PROOF

Theorem 1 Let f have an essential singularity at z_0 and let U be any (arbitrarily small) deleted neighborhood of z_0. Then, for all $w \in C$, except perhaps one value, the equation $f(z) = w$ has infinitely many solutions z in U.

Theorem 1 actually belongs in a more advanced course. However, we can easily prove a simple version.

Theorem 2 Let f have an essential singularity at z_0 and let $w \in C$. Then there exist $z_1, z_2, z_3, \ldots\ldots, z_n \to z_0$, such that $f(z) \to w$.

Proof If the assertion were false, there would be a deleted neighborhood U of z_0 and an $\varepsilon > 0$ such that $|f(z) - w| \geq \varepsilon$ for all $z \in U$. Let $g(z) = \frac{1}{f(z)-w}$. Thus on U, g is analytic, and since $g(z)$ is bounded on $U(|g(z)| \leq \varepsilon^{-1})$£$z_0$ is removable singularity by Lemma 1(i). Let k be the order of the zero of g at z_0(set k=0 if $g(z_0) \neq 0$). (The order must be finite because otherwise, as mentioned previously, by the Taylor Theorem, g would be zero in a neighborhood of z_0, whereas g is 0 nowhere on U.) Thus $f(z) = w + \frac{1}{g(z)}$ is either analytic(if k=0) or has a pole of order k by **Lemma 2**. This conclusion contradicts our assumption that f has an essential singularity.

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