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Exp-Function Method for Duffing Equation and New Solutions of (2+1) Dimensional Dispersive Long Wave Equations

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Abstract: In this paper, the general solutions of the Duffing equation with third degree nonlinear term is obtain using the Exp-function method. Using the Duffing equation and its general solution, the new and general exact solution with free parameter and arbitrary functions of the (2+1) dimensional dispersive long wave equation are obtained. Setting free parameters as special values, hyperbolic as well as trigonometric function solutions are also derived. With the aid of symbolic computation, the Exp-function method serves as an effective tool in solving the nonlinear equations under study.

Key words: Exp-Function Method; Duffing Equation; Exact Solutions; Nonlinear Evolution Equations

1. INTRODUCTION

Nonlinear phenomena appear in large range of scientific fields, such as applied mathematics, physics, engineering problems, plasma physics, fluid mechanics, nonlinear optics, solid state physics, chemical kinetics, geochemistry etc. Therefore, the investigation of the exact solutions for nonlinear evolution equations (NLEEs) plays an important role in the study of nonlinear physical phenomena. Yet, solving nonlinear differential equations corresponding to the nonlinear problems are often complicated. Particularly, getting their explicit solutions is even more difficult. Up to present, a lot of new methods for solving nonlinear differential equations are developed, for example, the tanh-function method[1, 2], the extended tanh method[3], Hirota's bilinear method [4], Backlund transformation method [5], F-expansion method [6-9], sine-cosine method [10], Jacobian elliptic function method[11-13], homogeneous balance method[14], homotopy perturbation method[15-17], variational iteration method[18-21], Adomian decomposition method[22], auxiliary equation method[23-26] and so on. Recently, He and Wu[27] proposed a straightforwad and concise method called the exp-function method to obtain the generalized solitonary solutions and periodic solutions of NLEEs. The Exp-function method has also been successfully applied to many kinds of NLEEs[29-43], such as difference-differential equations, high-dimensional equations, variable-coefficient equations, discrete equations, etc. Generally speaking, exact solutions of NLEEs

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obtained by most of these methods are written as a polynomial in several elementary or special functions that satisfy a first-order ordinary differential equation called the sub-equation, for example, the Riccati's equation. Second or higher-order nonlinear differential equations have not been considered. It is obvious that the more the solutions of the sub-equation we find, the more the exact solutions of the considered NLEEs we may obtain. The aim of the present paper is to use the Exp-function method[27] to seek general solutions of the Duffing equation:

$$\frac{d^2 z}{d\xi^2} + \omega_0^2 z - \varepsilon z^3 = 0 \quad (1)$$

where ω_0 and ε are real parameters. Then Eq. (1) is employed as a new auxiliary equation and its general solutions are applied to find new exact solutions of the (2+1)-dimensional dispersive long wave equations:

$$u_{yt} + H_{xx} + \tfrac{1}{2}(u^2)_{xy} = 0 \quad (2)$$

$$H_t + (uH + u + u_{xy})_x = 0 \quad (3)$$

In the case of compatibility condition for a weak lax pair, Boiti et al. [44] first introduced Eqs. (2) and (3). A variational model of Eqs. (2) and (3) was found by He[11] using the semi-inverse method.

2. EXP-FUNCTION METHOD FOR EQ. (1):

Following the Exp-function method[27], we suppose that the solution of Eq. (1) can be expressed in the form:

$$z(\xi) = \frac{a_1 \exp(l\xi + m) + a_0 + a_{-1} \exp(-(l\xi + m))}{b_1 \exp(l\xi + m) + b_0 + b_{-1} \exp(-(l\xi + m))} \quad (4)$$

where $a_1, a_0, a_{-1}, b_1, b_0, b_{-1}, k$ and m are constants which are unknown and to be determined later.

Substituting Eq. (4) into Eq. (1) and equating the coefficients of all powers of $\exp[i(l\xi + \omega)]$ ($i = 0, \pm 1, \pm 2, \pm 3, \pm 4$) to zero yields a set of algebraic equations for $a_1, a_0, a_{-1}, b_1, b_0, b_{-1}, l$ and m. Solving the system of algebraic equations with the help of Maple 12, we find:

$$l = \pm\sqrt{2}\,\omega_0,\ m = m,\ a_{-1} = a_{-1},\ a_0 = a_0,\ a_1 = \frac{\varepsilon a_0^2 - \omega_0^2 b_0^2}{4\varepsilon a_{-1}},\ b_{-1} = \pm\frac{a_{-1}\sqrt{\varepsilon}}{\omega_0},\ b_0 = b_0,$$

$$b_1 = \pm\frac{\omega_0^2 b_0^2 - \varepsilon a_0^2}{4 a_{-1}\omega_0\sqrt{\varepsilon}} \quad (5)$$

$$l = \pm i\,\omega_0,\ m = m,\ a_{-1} = 0,\ a_0 = a_0,\ a_1 = 0,\ b_{-1} = b_{-1},\ b_0 = 0,\ b_1 = \frac{1}{8}\frac{\varepsilon a_0^2}{\omega_0^2 b_{-1}} \quad (6)$$

$$l = \pm\sqrt{2}\,\omega_0,\ m = m,\ a_{-1} = a_{-1},\ a_0 = a_0,\ a_1 = a_1,\ b_{-1} = \pm\frac{a_{-1}\sqrt{\varepsilon}}{\omega_0},\ b_0 = \pm\frac{a_0}{\omega_0}\sqrt{\varepsilon},$$

$$b_1 = \pm\frac{\sqrt{\varepsilon}\,a_1}{\omega_0} \quad (7)$$

Therefore, we obtain the following general solutions of Eq. (1) for (5) and (6) respectively:

$$z(\xi) = \mp\frac{\omega_0}{\sqrt{\varepsilon}} + \frac{4\omega_0 a_{-1}\{\sqrt{\varepsilon}\,a_0 \pm b_0\omega_0 + 2\sqrt{\varepsilon}\,a_{-1}\exp(\mp\sqrt{2}\,\omega_0\xi - m)\}}{\pm(\omega_0^2 b_0^2 - \varepsilon a_0^2)\exp(\pm\sqrt{2}\,\omega_0\xi + m) + 4\sqrt{\varepsilon}\,b_0\,\omega_0\,a_{-1} \pm 4\varepsilon a_{-1}^2\exp(\mp\sqrt{2}\,\omega_0\,\xi - m)} \tag{8}$$

$$z(\xi) = \frac{8\omega_0^2 a_0\, b_{-1}}{\varepsilon a_0^2\exp(\pm i\omega_0\,\xi + m) + 8\omega_0^2 b_{-1}^2\exp(\mp i\omega_0\xi - m)} \tag{9}$$

And for (7), we obtain the constant solution $z = \pm\frac{\omega_0}{\sqrt{\varepsilon}}$ of the equation (1).

Setting $a_0 = \frac{1}{\sqrt{\varepsilon}}$, $a_{-1} = \frac{1}{2\sqrt{\varepsilon}}$, $b_0 = \frac{\sqrt{2}}{\omega_0}$ and simplifying Eq. (8), we obtain

$$z(\xi) = \frac{\omega_0}{\sqrt{\varepsilon}}\frac{1 \mp \sinh(\sqrt{2}\,\omega_0\,\xi \pm m)}{\sqrt{2} \pm \cosh(\sqrt{2}\,\omega_0\xi \pm m)} \tag{10a}$$

or , $$z(\xi) = \frac{\omega_0}{\sqrt{\varepsilon}}\frac{\operatorname{csch}(\sqrt{2}\,\omega_0\,\xi \pm m) \mp 1)}{\sqrt{2}\operatorname{csch}(\sqrt{2}\,\omega_0\,\xi \pm m) \pm \coth(\sqrt{2}\,\omega_0\xi \pm m)} \tag{10b}$$

And simplifying Eq. (9), we obtain

$$z(\xi) = \pm\sqrt{\frac{2}{\varepsilon}}\,\omega_0\sec(\omega_0\,\xi \pm m) \tag{11}$$

where m is replaced by im and $b_{-1} = \frac{\sqrt{\varepsilon}\,a_0}{2\sqrt{2}\,\omega_0}$.

These are the exact solutions of the Eq. (1). We observe that equations (8) and (9) are the general exact solutions of the Duffing equation. The more important point is, if we use Eq. (1) and its general solutions (8) and (9), we can obtain new and general exact solutions of Eqs. (2) and (3). The solution (10b) (equivalent to Eq. (10a)) is the fractional form of csch and $\coth$ functions. They are useful to obtain singular travelling wave solutions with important physical significance and solution (11) is also useful to obtain singular travelling wave solutions.

3. EXACT SOLUTIONS OF EQS. (2) AND (3):

Using the homogeneous balance method, we suppose that Eqs. (2) and (3) have the following formal solutions:

$$u = a_0(y,t) + a_1(y,t)\,z(\xi) \tag{12}$$

$$H = b_0(y,t) + b_1(y,t)\,z(\xi) + b_2(y,t)\,z^2(\xi) \tag{13}$$

where $z(\xi)$ satisfies Eq. (1), $\xi = l\,x + \eta(y,t)$, $a_0(y,t)$, $a_1(y,t)$, $b_0(y,t)$, $b_1(y,t)$, $b_2(y,t)$ and $\eta(y,t)$ are functions of y and t to be determined later, l is a nonzero constant.

Substituting Eqs. (12) and (13) together with Eq. (1), into Eqs. (2) and (3), the left-hand sides of Eqs. (2) and (3) are converted into two polynomials of $z^{/i}(\xi)\,z^{j}(\xi)(i=0,1;\ j=0,1,2,...)$, then putting each coefficient to zero, we get a set of over-determined partial differential equations for $a_0(y,t)$, $a_1(y,t)$, $b_0(y,t)$, $b_1(y,t)$, $b_2(y,t)$ and $\eta(y,t)$ as follows:

$$\frac{\partial^2}{\partial y\,\partial t}a_0(y,t)=0$$

$$2\,a_1^2(y,t)\,l\frac{\partial}{\partial y}\eta(y,t)+4\,b_2(y,t)\,l^2+2\,a_1(y,t)\,l\frac{\partial}{\partial y}a_1(y,t)=0$$

$$\frac{\partial^2}{\partial y\,\partial t}a_1(y,t)-a_0(y,t)\,a_1(y,t)\,l\frac{\partial}{\partial y}\eta(y,t)-b_1(y,t)\,l^2-a_1(y,t)\frac{\partial}{\partial y}\eta(y,t)\frac{\partial}{\partial t}\eta(y,t)=0$$

$$a_0(y,t)\frac{\partial}{\partial y}a_1(y,t)\,l+\frac{\partial}{\partial t}a_1(y,t)\frac{\partial}{\partial y}\eta(y,t)+\frac{\partial}{\partial y}a_1(y,t)\frac{\partial}{\partial t}\eta(y,t)+a_1(y,t)\,l\frac{\partial}{\partial y}a_0(y,t)+a_1(y,t)\frac{\partial^2}{\partial y\,\partial t}\eta(y,t)=0$$

$$b_1(y,t)\,l^2+a_0(y,t)\,a_1(y,t)\,l\frac{\partial}{\partial y}\eta(y,t)+a_1(y,t)\frac{\partial}{\partial y}\eta(y,t)\frac{\partial}{\partial t}\eta(y,t)=0$$

$$2\,b_2(y,t)\,l^2+a_1^2(y,t)\,l\frac{\partial}{\partial y}\eta(y,t)=0$$

$$\frac{\partial}{\partial y}a_1(y,t)\,l^2=0$$

$$\frac{\partial}{\partial t}b_2(y,t)=0$$

$$3\,a_1(y,t)\,l\,b_2(y,t)+3\,a_1(y,t)\,l^2\frac{\partial}{\partial y}\eta(y,t)=0$$

$$-\frac{\partial}{\partial y}a_1(y,t)\,l^2+\frac{\partial}{\partial t}b_1(y,t)=0$$

$$2\,b_2(y,t)\frac{\partial}{\partial t}\eta(y,t)+2\,a_0(y,t)\,b_2(y,t)\,l+2\,a_1(y,t)\,l\,b_1(y,t)=0$$

$$\frac{\partial}{\partial t}b_0(y,t)=0$$

$$-a_1(y,t)\,l^2\frac{\partial}{\partial y}\eta(y,t)+b_1(y,t)\frac{\partial}{\partial t}\eta(y,t)+a_1(y,t)\,l+a_0(y,t)\,b_1(y,t)\,l+a_1(y,t)\,b_0(y,t)\,l=0$$

Solving the set of over-determined partial differential equations by the use of Maple 12, we obtain

$$l = l \quad , \quad a_0(y,t) = g'(t) \quad , \quad a_1(y,t) = \sqrt{2}\,l \quad , \quad b_0(y,t) = -1 - f(y) \quad , \quad b_1(y,t) = 0 \quad ,$$
$$b_2(y,t) = f(y) \quad , \quad \eta(y,t) = \int f(y)\,dy + g(t) \tag{14}$$

where $f(y)$ and $g(t)$ are arbitrary functions of y and t respectively, and $g'(t) = \frac{d}{dt} g(t)$.

Employing solution (8) and from Eqs. (12)-(14), we obtain the following exact solutions of Eqs. (2) and (3):

$$u = g'(t) + \sqrt{2}\,l\left[\mp\frac{\omega_0}{\sqrt{\varepsilon}} + \frac{4\omega_0 a_{-1}\{\sqrt{\varepsilon}\,a_0 \pm b_0\omega_0 + 2\sqrt{\varepsilon}\,a_{-1}\exp(\mp\sqrt{2}\,\omega_0\xi - m)\}}{\pm(\omega_0^2 b_0^2 - \varepsilon a_0^2)\exp(\pm\sqrt{2}\,\omega_0\xi + m) + 4\sqrt{\varepsilon}\,b_0\,\omega_0\,a_{-1} \pm 4\varepsilon a_{-1}^2\exp(\mp\sqrt{2}\,\omega_0\,\xi - m)}\right] \tag{15}$$

$$H = -1 - f(y) + f(y)\left[\mp\frac{\omega_0}{\sqrt{\varepsilon}} + \frac{4\omega_0 a_{-1}\{\sqrt{\varepsilon}\,a_0 \pm b_0\omega_0 + 2\sqrt{\varepsilon}\,a_{-1}\exp(\mp\sqrt{2}\,\omega_0\xi - m)\}}{\pm(\omega_0^2 b_0^2 - \varepsilon a_0^2)\exp(\pm\sqrt{2}\,\omega_0\xi + m) + 4\sqrt{\varepsilon}\,b_0\,\omega_0\,a_{-1} \pm 4\varepsilon a_{-1}^2\exp(\mp\sqrt{2}\,\omega_0\,\xi - m)}\right]^2 \tag{16}$$

where $\xi = l\,x + \int f(y)\,dy + g(t)$.

Using solution (9) and from Eqs. (12)-(14), we obtain the following exact solutions of Eqs. (2) and (3):

$$u = g'(t) + \sqrt{2}\,l\,\frac{8\omega_0^2 a_0\,b_{-1}}{\varepsilon a_0^2\exp(\pm i\omega_0\,\xi + m) + 8\omega_0^2 b_{-1}^2\exp(\mp i\omega_0\xi - m)} \tag{17}$$

$$H = -1 - f(y) + f(y)\left[\frac{8\omega_0^2 a_0\,b_{-1}}{\varepsilon a_0^2\exp(\pm i\omega_0\,\xi + m) + 8\omega_0^2 b_{-1}^2\exp(\mp i\omega_0\xi - m)}\right]^2 \tag{18}$$

where $\xi = l\,x + \int f(y)\,dy + g(t)$.

If we set $a_0 = \frac{1}{\sqrt{\varepsilon}}$, $b_0 = \frac{\sqrt{2}}{\omega_0}$, $a_{-1} = \frac{1}{2\sqrt{\varepsilon}}$ and simplify then the solution (15) and (16) become

$$u = g'(t) + \frac{\sqrt{2}}{\sqrt{\varepsilon}}\,l\,\omega_0\,\frac{1 \mp \sinh(\sqrt{2}\,\omega_0\,\xi \pm m)}{\sqrt{2} \pm \cosh(\sqrt{2}\,\omega_0\xi \pm m)} \tag{19}$$

$$H = -1 - f(y) + \frac{\omega_0^2}{\varepsilon}\,f(y)\left[\frac{1 \mp \sinh(\sqrt{2}\,\omega_0\,\xi \pm m)}{\sqrt{2} \pm \cosh(\sqrt{2}\,\omega_0\xi \pm m)}\right]^2 \tag{20}$$

And simplifying Eqs. (17) and (18), we obtain

$$u = g'(t) \pm \frac{2l\,\omega_0}{\sqrt{\varepsilon}}\sec(\omega_0\,\xi \pm m) \tag{21}$$

$$H = -1 - f(y) + \frac{2}{\varepsilon}\omega_0^2 f(y)\sec^2(\omega_0 \xi \pm m) \tag{22}$$

We have checked the solutions (8)-(9) and (15)-(22) with the help of Maple 12 by putting them into the original equations and found that they satisfy the Eqs. (1)-(3). To the best of our knowledge, solutions (15)-(16) and (17)-(18) are new and have not been found in the literature.

4. ZHANG ET AL. [28] SOLUTIONS

Zhang et al.[28] also investigated the solutions of Eqs. (2) and (3). They have used the Recatti's equation as auxiliary equation and found the solutions as follows:

$$u = \pm\frac{bk\sqrt{c}}{2c} - \frac{g'(t)}{k} \pm 4ak\sqrt{c}\frac{sech(\sqrt{a}\xi + \xi_{01})}{\sqrt{(b^2-4ac)} \mp b\,sech(\sqrt{a}\xi + \xi_{01})} \tag{23}$$

$$H = -1 + \frac{k(b^2-4ac)f(y)}{4c} \mp 2abk\,f(y)\frac{sech(\sqrt{a}\xi + \xi_{01})}{\sqrt{(b^2-4ac)} \mp b\,sech(\sqrt{a}\xi + \xi_{01})}$$

$$-8a^2ck\,f(y)\frac{sech^2(\sqrt{a}\xi + \xi_{01})}{[\sqrt{(b^2-4ac)} \mp b\,sech(\sqrt{a}\xi + \xi_{01})]^2} \tag{24}$$

when $b^2 - 4ac > 0$ and $\xi = kx + \int f(y)dy + g(t)$

And

$$u = \pm\frac{bk\sqrt{c}}{2c} - \frac{g'(t)}{k} \pm 4ak\sqrt{c}\frac{\csc h(\sqrt{a}\xi + \xi_{03})}{\sqrt{(4ac-b^2)} \mp b\csc h(\sqrt{a}\xi + \xi_{03})} \tag{25}$$

$$H = -1 + \frac{k(b^2-4ac)f(y)}{4c} \mp 2abk\,f(y)\frac{\csc h(\sqrt{a}\xi + \xi_{03})}{\sqrt{(4ac-b^2)} \mp b\csc h(\sqrt{a}\xi + \xi_{03})}$$

$$-8a^2ck\,f(y)\frac{\csc h^2(\sqrt{a}\xi + \xi_{03})}{[\sqrt{(4ac-b^2)} \mp b\csc h(\sqrt{a}\xi + \xi_{03})]^2} \tag{26}$$

when $b^2 - 4ac < 0$ and $\xi = kx + \int f(y)dy + g(t)$.

From Eqs. (23)-(26), we observe that no choice of a, b and c yield the solutions (19)-(22). To identify the distinctness of two solutions, the simple and powerful tool is to plot the graphs of the solutions. The solutions having the same graphs are usually equivalent. Some graphs of solutions (19)-(26) are given below:

Fig. 1 and Fig. 2 are obtained from solutions (19) and (20) for u and H respectively when $s = 1, \omega_0 = 1$, $l = 1, m = \pi/2, g(t) = t$, $f(y) = 1$, and $t = 0$.

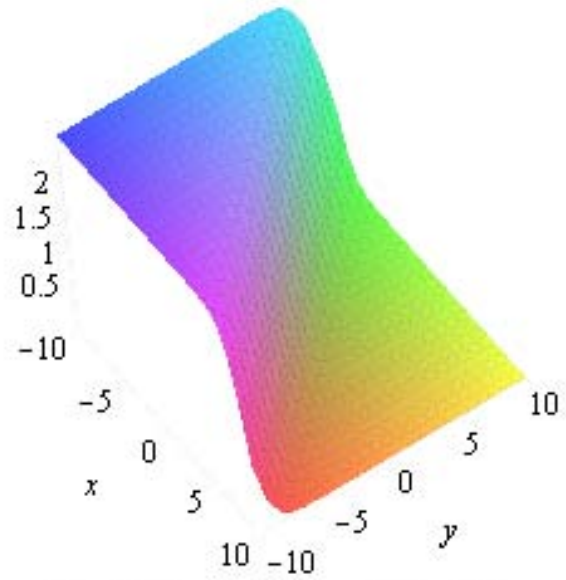

Fig. 1: Obtained from solution (19).

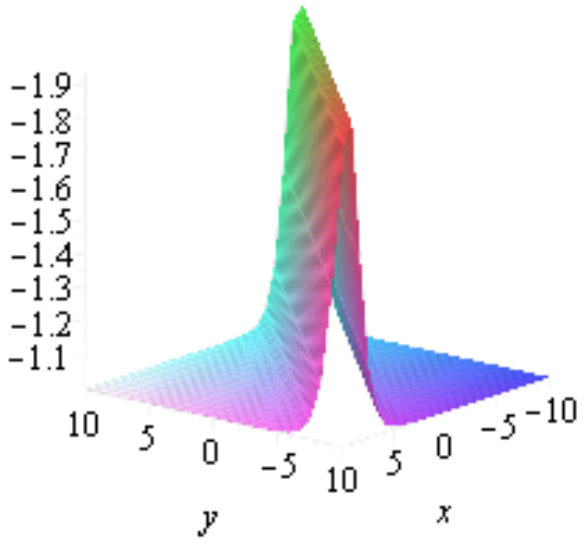

Fig. 2: Obtained from solution (20).

Fig. 3 and Fig. 4 are obtained from solutions (21) and (22) for u and H respectively when $s = 1, \omega_0 = 1, l = 1, m = 0, g(t) = t, f(y) = 1,$ and $t = 0.$

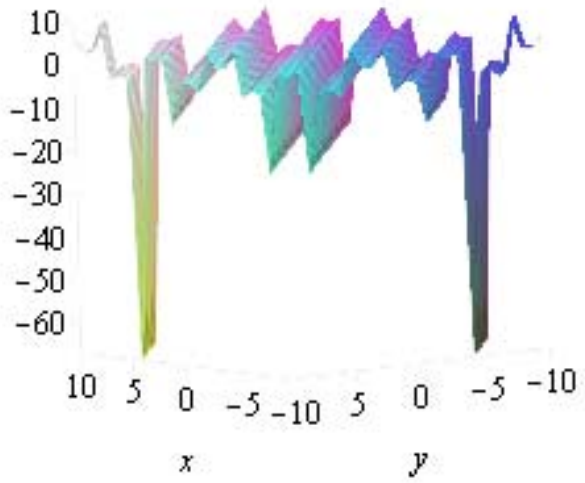

Fig. 3: Obtained from solution (21).

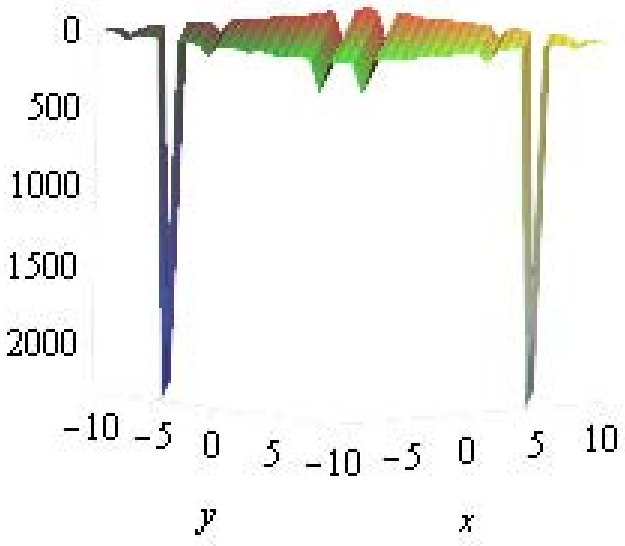

Fig. 4: Obtained from solution (22)

Fig. 5 and Fig. 6 are obtained from solutions (23) and (24) for u and H respectively when $a = 1, b = 3,$ c$= 2, k = 1, \xi_{01} = \pi/2, g(t) = t, f(y) = 1,$ and $t = 0.$

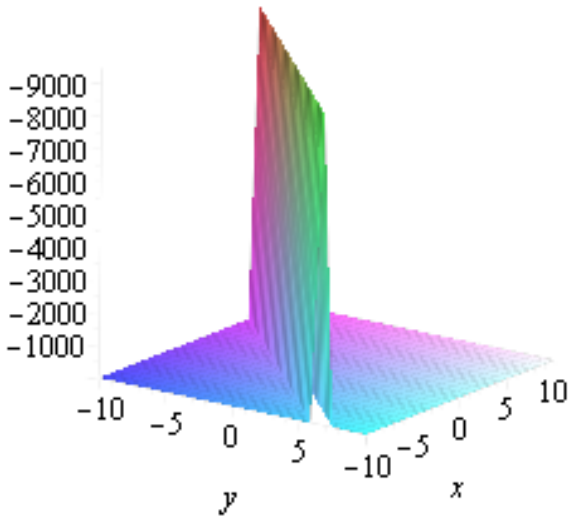

Fig. 5: Obtained from Zhang et al. solution (23)

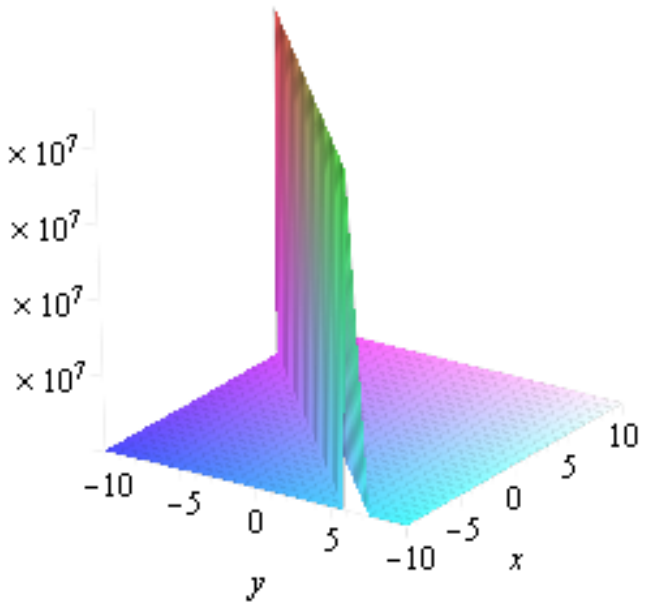

Fig. 6: Obtained from Zhang et al. solution (24)

Fig. 7 and Fig. 8 are obtained from solutions (23) and (24) for u and H respectively when $a = b = c = k = 1, \xi_{03} = \pi/2, g(t) = t, f(y) = 1,$ and $t = 0$.

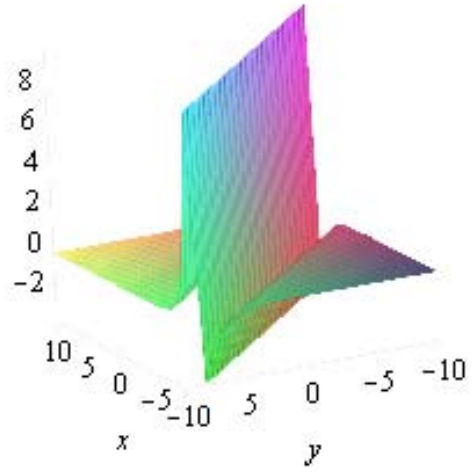

Fig. 7: Obtained from Zhang et al. solution (25)

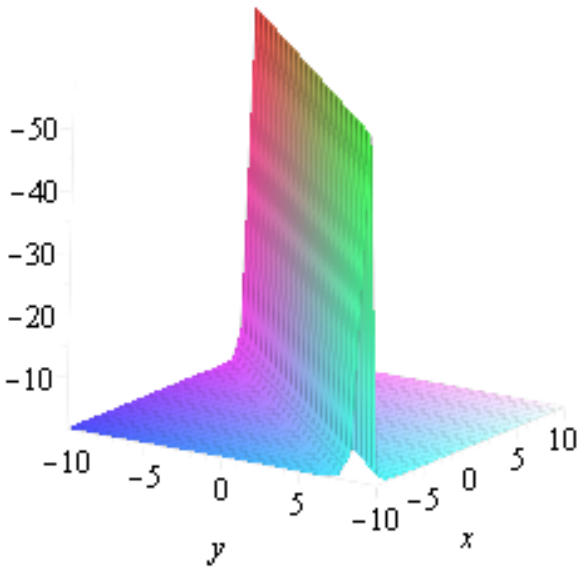

Fig. 8: Obtained from Zhang et al. solution (26)

It is seen, that the figures obtained from solutions (19)-(22) are different from the figures obtained from solutions (23)-(26).

5. CONCLUSION

Based on the exact solutions of the Duffing equation and its general solutions obtained by the Exp-function method, some new exact solutions with free parameters and arbitrary functions of the

(2+1)-dimensional dispersive long wave equations are obtained, from which some hyperbolic and trigonometric function solutions are also derived when setting the free parameters as special values. Solutions involving free parameters and arbitrary functions have rich local structures and are important for the explanation of physical phenomena. The presented method can also be applied to other NLEEs.

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