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Pricing Vulnerable Options under Stochastic Asset and Liability

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Abstract: In this paper pricing for vulnerable options is investigated. The discussed payoff function mainly derives from the Klein and the Ammann credit risk frameworks. Three stochastic processes, namely the underlying stock price, the asset value of the option writer, and the liability value of the option writer, are suitably modeled. Under the suggested payoff function, closed-form solutions for vulnerable European options are derived; moreover, adapting the Rubinstein's approach, a general binomial pyramid algorithm for vulnerable options pricing is constructed.
Key words: Credit Risk; Vulnerable Option Pricing; Binomial Pyramid Algorithm

1. INTRODUCTION

The study of options pricing subject to counterparty credit risk originated with Black and Scholes [2], and Merton [8], who took the first steps by investigating the credit risk models. Merton [8] developed an analytical framework, focusing on the case of defaulted debt instruments with a finite maturity date, assuming that the default might occur only at the expiration date. Hereafter, various methods for vulnerable options pricing were proposed; these can basically be divided into two categories: structural models and reduced-form models. In this paper, we will centralize attentions on the former models.

The structural model approaches focused on the evolution of the firm value to determine the default and recovery rate. Most of these models assumed that the credit event is the consequence of a firm's default, and that the default time-point is typically specified as the first moment at which the firm's asset value reaches a specific threshold boundary. Major investigation within these firm value models is to characterize the evolution of the firm's value, as well as the firm's capital structure; related papers include those of Merton [8], Johnson and Stulz [4], Klein [5], Klein and Inglis [6], and Ammann [1].

Following the structural model approach, the goal of this paper is to price vulnerable options by considering three stochastic processes: the underlying stock price, and the asset value, and the liability value of the option writer. Under a specified payoff function, both analytical and numerical solutions are investigated. The rest of this paper is organized as follows. Section 2 reviews certain credit risk models. Section 3 presents the discussion of the payoff function and the corresponding closed-form solution for vulnerable European options. Section 4 develops the binomial pyramid numerical algorithm for pricing

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vulnerable options. Finally, numerical evaluations are illustrated in Section 5, with Section 6 providing the conclusions of this paper.

2. REVIEW OF CREDIT RISK MODELS

The following notations will be employed throughout the financial market models discussed: at time u, S_u denotes the risky underlying stock price, V_u denotes the asset value, and D_u denotes the liability value of the option writer; r denotes the constant riskless interest rate and K denotes the strike price of the option contract; t denotes the present time point and T denotes the maturity date of the option contract. $N_m(\mu, \Lambda)$ denotes the *m*-variate normal distribution, with mean vector μ and variance matrix Λ, while $\Phi_m(\cdot)$ denotes the standard normal cumulative distribution function.

A continuous trading economy with trading interval $[t, T]$ is considered. The discounted factor is given by $e^{-r(T-t)}$ under the deterministic interest rates assumption. The financial market is assumed to be frictionless, arbitrage-free, and complete, so that all securities are perfectly divisible; there are no short-sale restrictions, transaction costs, or taxes. Furthermore, the stock pays no dividends during the period considered.

The valuation model concerned with default risk on option writers was first proposed by Johnson and Stulz [4], and assumes that the option itself is the only liability of the option writer. Without a consideration of deadweight costs, a payoff function for a vulnerable European call option is defined as:

$$C_T = (S_T - K)^+ \cdot \left[1_{\{V_T \geq S_T - K\}} + 1_{\{V_T < S_T - K\}} \frac{V_T}{(S_T - K)} \right], \qquad (1)$$

where $a^+ = \max\{0, a\}$ and $1_{\{\cdot\}}$ is an indicator function.

Klein [5] modified this model by permitting the existence of other liability in the capital structure of the option writer, as well as by introducing the concept of deadweight costs into his credit risk model. The payoff function for a vulnerable European call option is thus defined as:

$$C_T = (S_T - K)^+ \cdot \left[1_{\{V_T \geq D^*\}} + 1_{\{V_T < D^*\}} \frac{(1-\alpha)V_T}{D_t} \right]. \qquad (2)$$

The parameter α, expressed as a percentage of the option writer's asset value with $0 \leq \alpha \leq 1$, represents the deadweight costs associated with the default event. It includes the direct cost of bankruptcy of the reorganization process, as well as the indirect effects of distress on the business operations of the option writer. The parameter D^*, a fixed default boundary, could be strictly less than D_T due to the possibility of continuing in operations even when V_T is less than D_T. Both α and D^* are exogenously known constants.

Later, a more comprehensive payoff function for a vulnerable European call option was introduced by Klein and Inglis [6]

$$C_T = (S_T - K)^+ \cdot \left[1_{\{V_T \geq S_T - K + D^*\}} + 1_{\{V_T < S_T - K + D^*\}} \frac{(1-\alpha)V_T}{(S_T - K) + D^*} \right]. \qquad (3)$$

Nonetheless, with this Klein and Inglis[6] were not presenting a closed-form pricing formula, but were instead providing an approximating evaluation solution.

Ammann[1], on the other hand, developed the credit risk model of Johnson and Stulz[4] by extending the dynamic of the option writer's liability, D_T, to become a stochastic process. The payoff function for a vulnerable European call option is then:

$$C_T = (S_T - K)^+ \cdot \left[1_{\{V_T \geq D_T\}} + 1_{\{V_T < D_T\}} \delta_T \right]. \qquad (4)$$

The recovery rate, δ_T, itself follows a stochastic process, and can be exogenously estimated by using other econometric models. Meanwhile, a closed-form solution to the vulnerable European options (4) had been obtained.

3. THE PROPOSED PRICING MODEL

In this section, a closed-form formula for proposed vulnerable European options will be derived. The stochastic processes of S_t, V_t and D_t under a scheme analogous to the Black-Scholes model setting are assumed. The following stochastic differential equations are assumed:

$$\begin{cases} \dfrac{dS_u}{S_u} = \left(\mu_S du + \sigma_S dW_u^S\right) \\ \dfrac{dV_u}{V_u} = \left(\mu_V du + \sigma_V dW_u^V\right), \qquad \text{for } u \in [t,T], \\ \dfrac{dD_u}{D_u} = \left(\mu_D du + \sigma_D dW_u^D\right) \end{cases}$$

where μ_S, μ_V, and μ_D are constant drift coefficients; σ_S, σ_V, and σ_D are constant diffusion coefficients; and $W_u = (W_u^S, W_u^V,\ W_u^D)$ is a three-dimensional Wiener process under the no-arbitrage martingale measure Q, satisfying $E^Q\left(W_u^X\right) = 0$ for $X \in \{S,V,D\}$ and $E^Q\left(W_u^X, W_u^Y\right) = u\rho_{XY}$, $\sigma_{XY} = \rho_{XY}\sigma_X\sigma_Y$ for $X, Y \in \{S,V,D\}$.

It follows that $(\ln S_T, \ln V_T, \ln D_T)$ are normally distributed and denoted by

$$(\ln S_T, \ln V_T, \ln D_T) \overset{d}{\sim} N_3(\mu, \Lambda),$$

with mean vector μ, and covariance matrix Λ, given respectively by

$$\mu = \begin{bmatrix} \ln S_t + \left(r - \sigma_S^2/2\right)(T-t) \\ \ln V_t + \left(r - \sigma_V^2/2\right)(T-t) \\ \ln D_t + \left(r - \sigma_D^2/2\right)(T-t) \end{bmatrix}, \text{ and } \Lambda = (T-t)\begin{bmatrix} \sigma_S^2 & \sigma_{SV} & \sigma_{SD} \\ \sigma_{SV} & \sigma_V^2 & \sigma_{VD} \\ \sigma_{SD} & \sigma_{VD} & \sigma_D^2 \end{bmatrix}.$$

3.1 The Payoff Function

Allowing for flexibility of the default ratio, as well as a consideration of deadweight costs, a payoff function for a vulnerable European call option is proposed, through modification of Ammann's credit risk model, as:

$$C_T = \begin{cases} (S_T - K) & \text{, if } S_T - K > 0 \text{ and } \dfrac{V_T}{D_T} \geq d^* \\ (S_T - K)\cdot(1-\alpha)\dfrac{V_T}{D_T} & \text{, if } S_T - K > 0 \text{ and } \dfrac{V_T}{D_T} < d^* \\ 0 & \text{, if } S_T - K \leq 0 \end{cases}$$

$$= (S_T - K)^+ \cdot \left[1_{\left\{\frac{V_T}{D_T} \geq d^*\right\}} + (1-\alpha)\frac{V_T}{D_T} \cdot 1_{\left\{\frac{V_T}{D_T} < d^*\right\}} \right]. \quad (5)$$

Again, the parameter α represents the deadweight costs associated with the default event. The parameter d^*, a constant default boundary ratio, could strictly speaking, be less than 1, due to the possibility of a firm continuing to operate, even when V_T is less than D_T. Both α and d^* are given in advance. The relationship between model (5) and the reviewed models in Section 2 are summarized as follows:

1. If set $D_T = (S_T - K)$, $d^* = 1$, and $\alpha = 0$, then model (5) is reduced to the Johnson and Stulz[4] model presented by formula (1).

2. If set $D_T = D_t$ and $d^* = D^*/D_t$, then model (5) is reduced to the Klein[5] model presented by formula (2).

3. If set $D_T = (S_T - K) + D^*$ and $d^* = 1$, then model (5) is reduced to the Klein and Inglis[6] model presented by formula (3).

4. If set $d^* = 1$, $\delta_T = V_T / D_T$, and $\alpha = 0$, then model (5) is reduced to the Ammann[1] model presented by formula (4).

Therefore, the payoff function suggested in this paper is a generalized version of credit risk models that has been presented before. According to the Risk-Neutral Valuation Principle, the current time t arbitrage price of the vulnerable European call option is the deflated expected value from time T under the martingale measure Q. Thus the arbitrage pricing process, with the payoff provided by formula (5), is expressed as:

$$\Pi_C(t) = e^{-r(T-t)} \cdot E_t^Q \left[(S_T - K)^+ \cdot \left(1_{\left\{\frac{V_T}{D_T} \geq d^*\right\}} + (1-\alpha)\cdot 1_{\left\{\frac{V_T}{D_T} < d^*\right\}} \cdot \frac{V_T}{D_T} \right) \right]. \quad (6)$$

3.2 Closed-Form Formula for Vulnerable European Options

To simplify the derivation of the closed-form solution to formula (6), algebraic calculation is required: Let $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -1\end{bmatrix}$, $X^T=[\ln S_T \;\; \ln V_T \;\; \ln D_T]$, and $\delta_T=V_T/D_T$, which leads to

$$AX=(\ln S_T,\ln\delta_T)=(\ln S_T,\ln V_T-\ln D_T)\overset{d}{\sim}N_2(\mu_*,\Sigma), \tag{7}$$

where $\mu_*=\begin{pmatrix}\mu_S\\ \mu_\delta\end{pmatrix}=\begin{bmatrix}\ln S_t+(T-t)(r-\sigma_S^2/2)\\ \ln\delta_t-(T-t)(\sigma_V^2-\sigma_D^2)/2\end{bmatrix}$, $\Sigma=(T-t)\begin{bmatrix}\sigma_S^2 & \rho_{S\delta}\sigma_S\sigma_\delta\\ \rho_{S\delta}\sigma_S\sigma_\delta & \sigma_\delta^2\end{bmatrix}$,

$\sigma_\delta=\sqrt{\sigma_V^2+\sigma_D^2-2\rho_{VD}\sigma_V\sigma_D}$ and $\rho_{S\delta}=(\rho_{SV}\sigma_V-\rho_{SD}\sigma_D)/\sigma_\delta$.

Equations (5) and (6) can, thus, be respectively re-written as:

$$C_T=(S_T-K)^+\cdot\left[1_{\{\delta_T\geq d^*\}}+(1-\alpha)\delta_T\cdot 1_{\{\delta_T<d^*\}}\right], \tag{8}$$

$$\Pi_C(t)=e^{-r(T-t)}E_t^Q\left[(S_T-K)^+\cdot\left(1_{\{\delta_T\geq d^*\}}+(1-\alpha)\delta_T\cdot 1_{\{\delta_T<d^*\}}\right)\right].$$

These pre-conditions lead to the following theorem.

Theorem 1. (Arbitrage Pricing of Vulnerable European Options)

(a). The price of the stated vulnerable European call option at time t can be given as:

$$\begin{aligned}\Pi_C(t)&=S_t\Phi_2(a_1,a_2;\rho_{S\delta})-Ke^{-r(T-t)}\Phi_2(b_1,b_2;\rho_{S\delta})\\ &+S_t(1-\alpha)\delta_t\exp\left\{(\sigma_{S\delta}+\sigma_D^2-\sigma_{VD})(T-t)\right\}\Phi_2(c_1,c_2;-\rho_{S\delta})\\ &-K(1-\alpha)\delta_t\exp\left\{(\sigma_D^2-\sigma_{VD}-r)(T-t)\right\}\Phi_2(d_1,d_2;-\rho_{S\delta}),\end{aligned} \tag{9}$$

where $a_1=\dfrac{\ln(S_t/K)+(r+\sigma_S^2/2)(T-t)}{\sigma_S\sqrt{T-t}}$, $a_2=\dfrac{\ln(\delta_t/d^*)-(\sigma_V^2-\sigma_D^2-2\sigma_{S\delta})(T-t)/2}{\sigma_\delta\sqrt{T-t}}$,

$b_1=a_1-\sigma_S\sqrt{T-t}$, $b_2=a_2-\rho_{S\delta}\sigma_S\sqrt{T-t}$, $c_1=a_1+\rho_{S\delta}\sigma_\delta\sqrt{T-t}$, $c_2=-a_2-\sigma_\delta\sqrt{T-t}$, $d_1=b_1+\rho_{S\delta}\sigma_\delta\sqrt{T-t}$ and $d_2=-b_2-\sigma_\delta\sqrt{T-t}$.

(b). The price of the vulnerable European put option at time t is given by

$$\begin{aligned}\Pi_P(t)&=-S_t\Phi_2(-a_1,a_2;-\rho_{S\delta})+Ke^{-r(T-t)}\Phi_2(-b_1,b_2;-\rho_{S\delta})\\ &-(1-\alpha)S_t\delta_t\exp\left\{(\sigma_{S\delta}+\sigma_D^2-\sigma_{VD})(T-t)\right\}\Phi_2(-c_1,c_2;\rho_{S\delta})\end{aligned}$$

$$+K(1-\alpha)\delta_t \exp\left\{\left(\sigma_D^2-\sigma_{VD}-r\right)(T-t)\right\}\Phi_2\left(-d_1,d_2;\rho_{S\delta}\right),$$

with the payoff function defined by $P_T=(K-S_T)^+\cdot\left[1_{\{\delta_T\geq d^*\}}+(1-\alpha)\delta_T\cdot 1_{\{\delta_T<d^*\}}\right]$.

Proof: For the proof of (9), refer to Appendix A.

A put-call parity for a vulnerable European option can be directly derived from results of Theorem 1, and describes the relationship between the value of a call option and a put option, with the same underlying asset value, exercise price and expiration date.

Theorem 2. The resultant put-call parity for a vulnerable European option is:

$$\Pi_C(t)+\theta_1 Ke^{-r(T-t)}=\Pi_P(t)+\theta_2 S_t,$$

where $\theta_1=\Phi_1(b_2)+(1-\alpha)\delta_t \exp\left\{\left(\sigma_D^2-\sigma_{VD}\right)(T-t)\right\}\Phi_1(d_2)$, and

$$\theta_2=\Phi_1(a_2)+(1-\alpha)\delta_t \exp\left\{\left(\sigma_{S\delta}+\sigma_D^2-\sigma_{VD}\right)(T-t)\right\}\Phi_1(c_2).$$

The result of Theorem 2 implies that a long position of a vulnerable call option combined with a certain amount of cash, $\theta_1 Ke^{-r(T-t)}$, is equivalent to a long position of a vulnerable put option plus a long position in a stock with the size of θ_2. In other words, the payoff of the portfolio $\Pi_C(t)+\theta_1 Ke^{-r(T-t)}$ is equivalent to the payoff of the portfolio $\Pi_P(t)+\theta_2 S_t$, and each of the two portfolios can replicate the payoff pattern of the other. This provides a trading strategy to hedge each of the portfolios through position of the opposite option.

The delta hedge ratio of a stock option is the ratio of the change in the option value with respect to the change in the underlying stock price. The results of Theorem 1 lead to the following delta hedge ratios for vulnerable European options.

Theorem 3. The delta hedge ratio of a vulnerable European call (put) option is:

$$\Delta_C(t)=\frac{\partial\Pi_C(t)}{\partial S_t}=\Phi_2\left(a_1,a_2;\rho_{S\delta}\right)+(1-\alpha)\tau_t\delta_t\Phi_2\left(c_1,c_2;-\rho_{S\delta}\right)>0,$$

$$\Delta_P(t)=\frac{\partial\Pi_P(t)}{\partial S_t}=-\Phi_2\left(-a_1,a_2;-\rho_{S\delta}\right)-(1-\alpha)\tau_t\delta_t\Phi_2\left(-c_1,c_2;\rho_{S\delta}\right)<0,$$

where $\tau_t=\exp\left\{\left(\sigma_{S\delta}+\sigma_D^2-\sigma_{VD}\right)(T-t)\right\}$.

For the standard Black-Scholes[2] European options, the associated delta hedge ratio for a call option is $\Delta_C^{BS}(t)=\Phi_1(a_1)$, while for a put option it is $\Delta_P^{BS}(t)=-\Phi_1(-a_1)$, with a_1 provided by Theorem 1. After algebraic calculation, the following corollary demonstrates the delta hedge ratio relationships between the standard Black-Scholes model and the discussed vulnerable model.

Corollary 1. The relationships between delta hedge ratios are:

$$\Delta_C(t)-\Delta_C^{BS}(t)=-\Phi_2\left(a_1,-a_2;-\rho_{S\delta}\right)+(1-\alpha)\tau_t\delta_t\Phi_2\left(c_1,c_2;-\rho_{S\delta}\right),$$ and

$$\Delta_P(t)-\Delta_P^{BS}(t)=\Phi_2\left(-a_1,-a_2;\rho_{S\delta}\right)-(1-\alpha)\tau_t\delta_t\Phi_2\left(-c_1,c_2;\rho_{S\delta}\right).$$

4. BINOMIAL PYRAMID (BP) ALGORITHM

The numerical approach is especially useful for the pricing of American options. In this section, a pricing algorithm, derived through the construction of a joint bi-variate binomial lattice structure, called a BP algorithm, is discussed. Rubinstein [9] utilized a specified joint lattice structure with some equal transition probabilities to evaluate an option targeted on two risky underlying assets. In order to relax the equal probabilities assumptions, Ammann[1] imposed an independence assumption on the associated processes instead. A more general BP algorithm can be derived as follows.

4.1 Construction of the Binomial Pyramid Algorithms

Statement (7) can be represented as:

$$\left(\ln S_{t+\Delta_n}, \ln \delta_{t+\Delta_n}\right) \overset{d}{\sim} N_2\left(\mu_S^*, \mu_\delta^*, \sigma_S^2 \Delta_n, \sigma_\delta^2 \Delta_n, \rho_{S\delta}\right),$$

where $\mu_S^* = \ln S_t + \left(r - \sigma_S^2/2\right)\Delta_n$, $\mu_\delta^* = \ln \delta_t - \left(\sigma_V^2 - \sigma_D^2\right)\Delta_n/2$. In other words,

$$\begin{cases} S_{t+\Delta_n} = S_t \exp\left\{\left(r - \sigma_S^2/2\right)\Delta_n + \sigma_S W_{\Delta_n}^S\right\}, \\ \delta_{t+\Delta_n} = \delta_t \exp\left\{-\left(\sigma_V^2 - \sigma_D^2\right)\Delta_n/2 + \sigma_\delta W_{\Delta_n}^\delta\right\}. \end{cases}$$

Here $\mathbf{W} = (W_{\Delta_n}^S, W_{\Delta_n}^\delta)$ is a two-dimensional Wiener process under the measure Q, satisfying $E_t^Q\left(W_{\Delta_n}^S\right) = E_t^Q\left(W_{\Delta_n}^\delta\right) = 0$, and $E_t^Q\left(W_{\Delta_n}^S, W_{\Delta_n}^\delta\right) = \rho_{S\delta}\Delta_n$.

In order to model the correlated evolutions of S and δ, a correlated bi-variate binomial pyramid is constructed. Let the pair of initial values be represented by $\left(\delta_t, S_t\right)$; four different states, after a certain length of time has passed, are: $\left(\delta^u, S^u\right)$, $\left(\delta^d, S^u\right)$, $\left(\delta^d, S^d\right)$ and $\left(\delta^u, S^d\right)$, here $\delta^u = \delta_t u_\delta$, $S^u = S_t u_S$, $\delta^d = \delta_t d_\delta$ and $S^d = S_t d_S$, while δ^u (δ^d) and S^u (S^d) denote the values of the asset-to-debt ratio and the underlying stock price after an up-move (or down-move) with a jump size of u_δ (d_δ) and u_S (d_S), respectively. The corresponding restrictions for the no arbitrage assumption are $0 < d_\delta < 1 < u_\delta$ and $0 < d_S < 1 < u_S$. The four associated probabilities for the distinct states are defined as:

$$\begin{cases} p_1 = Q\left(\delta_{t+\Delta_n} = \delta^u, S_{t+\Delta_n} = S^u \mid \delta_t, S_t\right), \\ p_2 = Q\left(\delta_{t+\Delta_n} = \delta^d, S_{t+\Delta_n} = S^u \mid \delta_t, S_t\right), \\ p_3 = Q\left(\delta_{t+\Delta_n} = \delta^d, S_{t+\Delta_n} = S^d \mid \delta_t, S_t\right), \\ p_4 = Q\left(\delta_{t+\Delta_n} = \delta^u, S_{t+\Delta_n} = S^d \mid \delta_t, S_t\right), \end{cases} \quad \text{with } p_1 + p_2 + p_3 + p_4 = 1. \qquad (10)$$

To ensure that the convergence of the bi-variate binomial structure converges to the exact bi-variate normal distribution, and the recombining property, some defining equations are set up. Finally, $(u_\delta, d_\delta, u_S, d_S, p_1, p_2, p_3, p_4)$ are solved through the following equations:

$$\begin{cases} p_1 + p_2 + p_3 + p_4 = 1, \\ u_S d_S = 1, \\ u_\delta d_\delta = 1, \\ u_S(p_1 + p_2) + d_S(p_3 + p_4) = \exp\{r\Delta_n\}, \\ u_\delta(p_1 + p_4) + d_\delta(p_2 + p_3) = \exp\{-(\sigma_V^2 - \sigma_D^2 - \sigma_\delta^2)\Delta_n / 2\}, \\ u_S^2(p_1 + p_2) + d_S^2(p_3 + p_4) = \exp\{(2r + \sigma_S^2)\Delta_n\}, \\ u_\delta^2(p_1 + p_4) + d_\delta^2(p_2 + p_3) = \exp\{(\alpha_V + 3\alpha_D)\Delta_n\}, \\ u_S u_\delta p_1 + u_S d_\delta p_2 + d_S d_\delta p_3 + d_S u_\delta p_4 = \exp\{(r + \alpha_D + \sigma_{S\delta})\Delta_n\}. \end{cases} \tag{11}$$

Here $\alpha_V = \sigma_V^2 - \sigma_{VD}$ and $\alpha_D = \sigma_D^2 - \sigma_{VD}$.

Theorem 4. The solution to the parameters, under the restriction of the equations in (11), is:

$$u_\delta = \frac{1}{2}\Big[\exp(-\alpha_D\Delta_n) + \exp\{(\alpha_V + 2\alpha_D)\Delta_n\}$$

$$+\sqrt{\exp(-2\alpha_D\Delta_n) + \exp\{2(\alpha_V + 2\alpha_D)\Delta_n\} + 2\exp(\sigma_\delta^2\Delta_n) - 4}\Big],$$

$$u_S = \frac{1}{2}\left[\exp(-r\Delta_n) + \exp\{(r + \sigma_S^2)\Delta_n\} + \sqrt{\left[\exp(-r\Delta_n) + \exp\{(r + \sigma_S^2)\Delta_n\}\right]^2 - 4}\right]$$

$d_\delta u_\delta = 1$, $d_S u_S = 1$, $p_1 = 1 - \lambda_2 - p_4$, $p_2 = \lambda_2 - \lambda_1 + p_4$, and $p_3 = \lambda_1 - p_4$,

$$p_4 = \frac{\lambda_3 - u_\delta u_S - \lambda_2(d_\delta u_S - u_\delta u_S) - \lambda_1(d_\delta d_S - d_\delta u_S)}{u_\delta d_S - u_\delta u_S - d_\delta d_S + d_\delta u_S}, \text{ where}$$

$$\lambda_1 = \frac{u_S - \exp(r\Delta_n)}{u_S - d_S}, \quad \lambda_2 = \frac{u_\delta - \exp(\alpha_D\Delta_n)}{u_\delta - d_\delta}, \text{ and } \lambda_3 = \exp\{(r + \alpha_D + \sigma_{S\delta})\Delta_n\}.$$

Moreover, by ignoring higher order terms of Δ_n, approximations of u_δ and u_S, expressed as $u_\delta = \exp(\sigma_\delta\sqrt{\Delta_n})$ and $u_S = \exp(\sigma_S\sqrt{\Delta_n})$, are respectively obtained.

Proof: See Appendix B.

With the parameters derived from Theorem 4, the Binomial Pyramid (BP) algorithms can be developed. Assume that the initial values are given by $\delta_{0,0,0} = \delta_t$ and $S_{0,0,0} = S_t$. Let the information vector at each lattice node be represented by $\Theta_{m,i,j} = \left[\delta_{m,i,j}, S_{m,i,j}, \Gamma_{m,i,j}\right]$, for $0 \le m \le k_n$, and $0 \le i, j \le m$, where $\Gamma_{m,i,j} = \Gamma\left(\delta_{m,i,j}, S_{m,i,j}\right)$ is a function of $\delta_{m,i,j}$ and $S_{m,i,j}$, depending on the assumption of the payoff function corresponding to the BP algorithm. According to the payoff function defined by (8), is given as:

$$\Gamma_{m,i,j} = \Gamma\left(\delta_{m,i,j}, S_{m,i,j}\right) = \left(S_{m,i,j} - K\right)^+ \cdot \left[1_{\left\{\delta_{m,i,j} \ge d^*\right\}} + (1-\alpha)\delta_{m,i,j} \cdot 1_{\left\{\delta_{m,i,j} < d^*\right\}}\right],$$

for $1 \le m \le k_n$, and $0 \le i, j \le m$. Assuming that there is no default event at the current time t, the initial intrinsic value is $\Gamma_{0,0,0} = \left(S_{0,0,0} - K\right)^+$.

After time passing, four lattice nodes, namely, $(1,0,0)$, $(1,1,0)$, $(1,0,1)$, and $(1,1,1)$, are expanded from the initial node $(0,0,0)$. The values of $\delta_{1,i,j}, S_{1,i,j}$ and $\Gamma_{1,i,j}$ are arranged in order: $\delta_{1,i,j} = \delta_t \cdot (u_\delta)^i (d_\delta)^{1-i}$, $S_{1,i,j} = S_t \cdot (u_S)^j (d_S)^{1-j}$, $\Gamma_{1,i,j} = \Gamma\left(\delta_{1,i,j}, S_{1,i,j}\right)$, for $0 \le i, j \le 1$, with the transition probabilities $\{p_l\}$ given by equation (10). The special case of a two-step binomial pyramid, with $k_n = n = 2$, is demonstrated in Figure 1. The value of $\Theta_{1,i,j}$, for $0 \le i, j \le 1$, is systematically constructed with the initial information vector $\Theta_{0,0,0} = [\delta_{0,0,0}, S_{0,0,0}, \Gamma_{0,0,0}]$. After a second time step increase, the subsequent information vectors at layer 2 are forward derived. For instance, $\Theta_{2,2,2} = \left[\delta_{0,0,0}(u_\delta)^2, S_{0,0,0}(u_S)^2, \Gamma\left(\delta_{0,0,0}(u_\delta)^2, S_{0,0,0}(u_S)^2\right)\right]$ with probability p_1^2.

The general forward iterative procedure is given by:

$$\begin{cases} \delta_{m,i,j} = \delta_t \cdot (u_\delta)^i (d_\delta)^{m-i}, \\ S_{m,i,j} = S_t \cdot (u_S)^j (d_S)^{m-j}, \\ \Gamma_{m,i,j} = \left(S_{m,i,j} - K\right)^+ \cdot \left[1_{\left\{\delta_{m,i,j} \ge d^*\right\}} + (1-\alpha)\delta_{m,i,j} \cdot 1_{\left\{\delta_{m,i,j} < d^*\right\}}\right], \end{cases} \tag{12}$$

with the transition probabilities

$$\begin{cases} p_1 = Q\left(\delta_{t+m\Delta_n} = \delta_{t+(m-1)\Delta_n} u_\delta, S_{t+m\Delta_n} = S_{t+(m-1)\Delta_n} u_S \mid \delta_{t+(m-1)\Delta_n}, S_{t+(m-1)\Delta_n}\right), \\ p_2 = Q\left(\delta_{t+m\Delta_n} = \delta_{t+(m-1)\Delta_n} d_\delta, S_{t+m\Delta_n} = S_{t+(m-1)\Delta_n} u_S \mid \delta_{t+(m-1)\Delta_n}, S_{t+(m-1)\Delta_n}\right), \\ p_3 = Q\left(\delta_{t+m\Delta_n} = \delta_{t+(m-1)\Delta_n} d_\delta, S_{t+m\Delta_n} = S_{t+(m-1)\Delta_n} d_S \mid \delta_{t+(m-1)\Delta_n}, S_{t+(m-1)\Delta_n}\right), \\ p_4 = Q\left(\delta_{t+m\Delta_n} = \delta_{t+(m-1)\Delta_n} u_\delta, S_{t+m\Delta_n} = S_{t+(m-1)\Delta_n} d_S \mid \delta_{t+(m-1)\Delta_n}, S_{t+(m-1)\Delta_n}\right), \end{cases}$$

for $1 \le m \le k_n$, and $0 \le i, j \le m$. The initial value is $\Gamma_{0,0,0} = (S_t - K)^+$. It is worth noting that the construction of the BP algorithm under discussion has the advantage that either the equal probabilities assumption of Rubinstein [9] or the independence assumption of Ammann [1] is relaxed, and that the associated eight parameters are well-defined and explicitly expressed in Theorem 4.

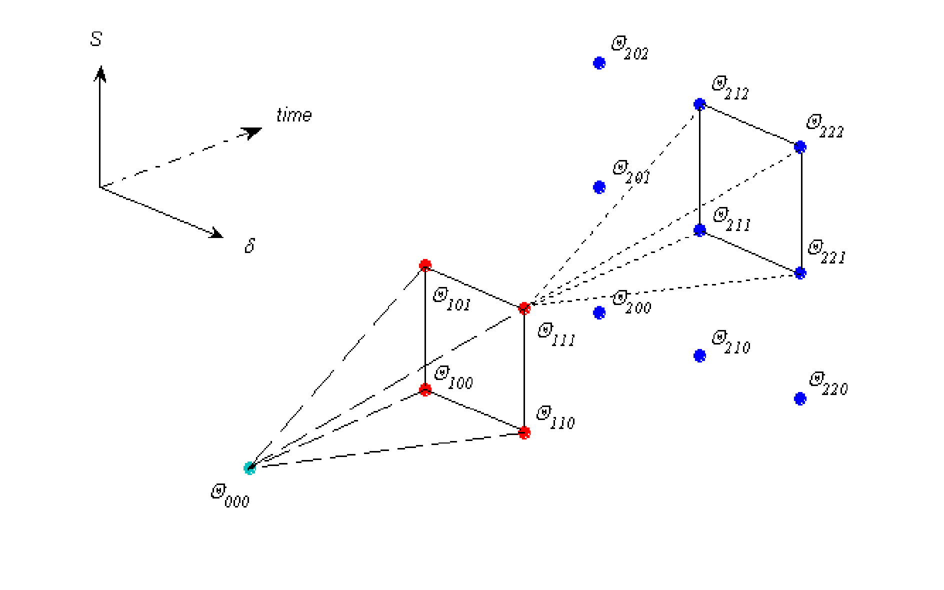

Figure 1: Forward Construction for a Two-step BP Structure

4.2 BP Algorithm for Pricing Vulnerable European Call Options

Let $F_{m,i,j}$ be the arbitrage value of the vulnerable European call option at the (m,i,j) node of the BP algorithm. Each center lattice induced from the associated four corner lattices, results in the following backward reduction steps:

$$F_{m,i,j} = e^{-r\Delta_n}\left(p_1 F_{m+1,i+1,j+1} + p_2 F_{m+1,i,j+1} + p_3 F_{m+1,i,j} + p_4 F_{m+1,i+1,j}\right), \quad (13)$$

for $0 \le m < k_n$, and $0 \le i, j \le m$, with initial conditions

$$F_{k_n,i,j} = \Gamma_{k_n,i,j}, \text{ for } 0 \le i, j \le k_n .$$

Beginning with the initial values $\{F_{k_n,i,j}\}_{i,j=0}^{k_n}$, and moving backward through every node of the BP tree, the arbitrage price of the vulnerable European call option at the current time point, $F_{0,0,0}$, is attained. The recursive procedure for the $k_n = n = 2$ special case, similar to Figure 1, with F replacing Θ, is summarized as follows:

1. Derive information vectors $\Theta_{1,i,j}$, for $0 \le i, j \le 1$, and then $\Theta_{2,i,j}$, for $0 \le i, j \le 2$.

2. Calculate intrinsic values $\Gamma_{2,i,j}$ from equation (12).

3. Obtain associated initial values $F_{2,i,j}$ from equation (12).

4. Repeat the backward processes of equation (13) in turn to derive $F_{1,i,j}$:

$$F_{1,1,1} = e^{-r\Delta_2}\left(p_1F_{2,2,2} + p_2F_{2,1,2} + p_3F_{2,1,1} + p_4F_{2,2,1}\right),$$

$$F_{1,0,1} = e^{-r\Delta_2}\left(p_1F_{2,1,2} + p_2F_{2,0,2} + p_3F_{2,0,1} + p_4F_{2,1,1}\right),$$

$$F_{1,1,0} = e^{-r\Delta_2}\left(p_1F_{2,2,1} + p_2F_{2,1,1} + p_3F_{2,1,0} + p_4F_{2,2,0}\right),$$

$$F_{1,0,0} = e^{-r\Delta_2}\left(p_1F_{2,1,1} + p_2F_{2,0,1} + p_3F_{2,0,0} + p_4F_{2,1,0}\right).$$

Finally, the current arbitrage price, $F_{0,0,0}$, is obtained:

$$F_{0,0,0} = e^{-r\Delta_2}\left(p_1F_{1,1,1} + p_2F_{1,0,1} + p_3F_{1,0,0} + p_4F_{1,1,0}\right).$$

Consider a special case in which the stock price and the asset-to-debt ratio are independent: the following relationships result:

$$p_1 \cdot p_3 = Q\left(\delta_{t+\Delta_n} = \delta^u, S_{t+\Delta_n} = S^u \mid \delta_t, S_t\right)\cdot Q\left(\delta_{t+\Delta_n} = \delta^d, S_{t+\Delta_n} = S^d \mid \delta_t, S_t\right)$$

$$= Q\left(\delta_{t+\Delta_n} = \delta^u, S_{t+\Delta_n} = S^d \mid \delta_t, S_t\right)\cdot Q\left(\delta_{t+\Delta_n} = \delta^d, S_{t+\Delta_n} = S^u \mid \delta_t, S_t\right) = p_4 \cdot p_2$$

By recursively using equation (13) under the assumption of independence, $p_1p_3 = p_2p_4$, Ammann[1] derived the following closed-form solution to vulnerable European call options:

$$\Pi_n(t) = e^{-r(T-t)}\sum_{i=0}^{k_n}\sum_{j=0}^{k_n}\left\{\binom{k_n}{i}\binom{k_n}{j}p_1^a p_2^b p_3^c p_4^d \Gamma_{k_n,i,j}\right\},$$

where $a = (i + j - k_n)^+$, $b = j - a$, $c = (k_n - i - j)^+$, $d = i - a$, $a + b + c + d = k_n$, and $\Gamma_{m,i,j}$ is given by equation (12). A further assumption of $p_1 = p_2 = p_3 = p_4 = 1/4$ results in

$$\Pi_n(t) = e^{-r(T-t)} \cdot \left(\frac{1}{4}\right)^{k_n} \cdot \sum_{i=0}^{k_n}\sum_{j=0}^{k_n}\left\{\binom{k_n}{i}\binom{k_n}{j}\Gamma_{k_n,i,j}\right\},$$

which can be easily solved if the payoff function $\Gamma_{m,i,j}$ is simple enough.

The above mentioned BP algorithm can be easily extended to price vulnerable American call options by taking the earlier exercise into consideration.

5. NUMERICAL ILLUSTRATIONS

A conditional binomial tree (CBT) algorithm for pricing European options has been proposed by Liu and Liu[7]. The CBT algorithm, essentially an extension of the CRR[3] model, has a dimension reduction effect. The CBT algorithm provides a rather simple and efficient technique by applying model (5), which calculates a one-dimensional normal cumulative distribution function (cdf) instead of a two-dimensional cdf in equation (9), or constructs a one-dimensional binomial tree instead of a complicated BP tree. Also, the dimension reduction effect avoids computational error caused by a numerical integral of a bi-variate normal cdf. A sketch of the CBT algorithm is provided in Appendix C. Moreover, results of the CBT algorithm actually converge to those stated in Theorem 1. For further details, please refer to Liu and Liu[7].

Table 1: Relative Percentage Error (%) for Vulnerable European Put Options

Case	Algorithm	$n=50$	$n=100$	$n=200$	$n=500$	$n=1000$
Base Case	CBT	0.5076	0.2541	0.1271	0.0508	0.0254
	B P	0.3864	0.3643	0.1217	0.0865	0.0700
$S_t=50$	CBT	0.8272	0.2697	0.0942	0.0611	0.0164
	B P	0.9367	0.1704	0.0989	0.0292	0.0235
$K=50$	CBT	0.1228	0.0397	0.0139	0.0092	0.0025
	B P	0.2571	0.0810	0.0199	0.0098	0.0062
$\delta_t=2$	CBT	0.5049	0.2551	0.1276	0.0511	0.0256
	B P	0.5049	0.2551	0.1276	0.0511	0.0256
$r=0.05$	CBT	0.5245	0.2626	0.1314	0.0526	0.0263
	B P	0.4036	0.3723	0.1259	0.0881	0.0707
$\sigma_S=1.0$	CBT	0.5030	0.2516	0.1257	0.0501	0.0249
	B P	0.3789	0.3643	0.1202	0.0866	0.0705
$\sigma_V=0.5$	CBT	0.5110	0.2559	0.1280	0.0513	0.0256
	B P	0.4074	0.2164	0.1174	0.0189	0.0443
$\sigma_D=0.3$	CBT	0.5094	0.2551	0.1276	0.0511	0.0256
	B P	0.5027	0.2520	0.1277	0.0533	0.0237
$\rho_{SV}=0$	CBT	0.5068	0.2538	0.1270	0.0509	0.0255
	B P	0.4926	0.2725	0.1270	0.0566	0.0325
$\rho_{SD}=0$	CBT	0.5096	0.2552	0.1277	0.0512	0.0256
	B P	0.1669	0.5734	0.1155	0.1540	0.1529
$\rho_{VD}=0.7$	CBT	0.5069	0.2541	0.1275	0.0514	0.0260
	B P	0.6067	0.2337	0.0795	0.0463	0.0228

1. Most calculations are under the Base Case unless otherwise noticed: For instance, the case of $S_t=50$, changes S_t from 40 to 50, and keeps the remaining unchanged.
2. The relative percentage error is computed by: $\left(|\tilde{x}-x|/x\right)\times 100\%$, where x is calculated from the closed-form solution, and $\tilde{x}$ is obtained by the designated numerical method.

In this section, numerical examples with CBT and BP algorithms are demonstrated for vulnerable European put options. The period number k_n is set as $k_n=n$, and a set of parameters called the *Base Case* is given as: $S_t=40$, $K=40$, $V_t=6$, $D_t=5$, $\delta_t=V_t/D_t=1.2$, $d^*=0.95$, $\alpha=0.3$, $r=0.02$, $T-t=0.25$, $\sigma_S=0.6$, $\sigma_V=0.3$, $\sigma_D=0.5$, $\rho_{SV}=0.4$, $\rho_{SD}=0.3$, and

$\rho_{VD} = 0.9$. Certain of the parameters are decided by S&P500 historical market data in order to provide the scenario of a real financial market.

Numerical comparisons with benchmark values derived from the closed-form solution given in Theorem 1 are provided in Table 1. The entries in Table 1 evaluate the relative percentage errors between the closed-form formula and the CBT algorithm, and between the closed-form formula and the BP algorithm, for the vulnerable European put option respectively. The period number n ranges from 50 to 1000, providing a demonstration of convergence patterns. In addition to the parameters of the *Base Case*, another ten numerical examples have also been provided by changing only one of the parameters. For instance, the case $S_t = 50$, changes S_t from 40 to 50, and keeps the remaining parameters unchanged.

The relative percentage errors listed in Table 1 decrease almost monotonically as the period number increases. Most of the relative percentage errors are less than 0.5% when the period number is larger than 100, are less than 0.1% when the period number is over 200, and are less than 0.03% when the period number equals 1000. Numerical examinations verify that the CBT algorithm for vulnerable European options is rather accurate, and that the algorithm is significantly simpler. The constructed BP algorithm is appropriate for use with vulnerable American options.

6. CONCLUSIONS

This paper discussed methods for pricing vulnerable options. Extending the Klein[5] and Ammann[1] credit risk models, the proposed payoff function considers three correlated stochastic processes: the underlying stock price, the option writer's asset value and the option writer's liability value. A closed-form solution and analytical results were derived under the discussed payoff function. Furthermore, the Binomial Pyramid (BP) algorithms were constructed as discrete time approximating procedures.

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APPENDIX A

Proof of formula (9)

Before proving Theorem 1, some preliminary results are presented without proof as follows:

A 1. $\ln\delta_T\,|_{S_T=s} \overset{d}{\sim} N_1\left(\alpha(s),\beta^2\right)$, where $\alpha(s)=\mu_\delta+\rho_{S\delta}\dfrac{\sigma_\delta}{\sigma_S}\left(\ln s-\mu_S\right)$, $\mu_\delta=\ln\delta_t-(T-t)\left(\sigma_V^2-\sigma_D^2\right)/2$, $\mu_S=\ln S_t+\left(r-\sigma_S^2/2\right)(T-t)$ and $\beta=\sigma_\delta\sqrt{\left(1-\rho_{S\delta}^2\right)(T-t)}$.

A 2. Let $\varphi_2(x,y;\rho)=\dfrac{1}{2\pi\sqrt{1-\rho^2}}\exp\left[-\dfrac{\left(x^2-2\rho xy+y^2\right)}{2\left(1-\rho^2\right)}\right]$,

$\Phi_2(x,y;\rho)=\int_{-\infty}^{x}\int_{-\infty}^{y}\varphi_2(u,v;\rho)dvdu$, and $\varphi_1(x)=\dfrac{1}{\sqrt{2\pi}}\exp\left(-\dfrac{x^2}{2}\right)$, then for any standard normal random variable Y ,

1. $\int_{-\infty}^{x}\Phi_1\left(\dfrac{y-z\rho}{\sqrt{1-\rho^2}}\right)\cdot\varphi_1(z)dz=\Phi_2(x,y;\rho)$.

2. $E\left\{\exp(aY)\cdot 1_{\{Y>y\}}\Phi_1\left(\dfrac{b+\rho Y}{\sqrt{1-\rho^2}}\right)\right\}=\exp\left(\dfrac{a^2}{2}\right)\Phi_2(a-y,b+\rho a;\rho)$.

A 3. $E_t^Q\left[1_{\left\{\delta_T\geq d^*\right\}}\middle|S_T=s\right]=\Phi_1\left(\dfrac{b_2+\rho_{S\delta}z}{\sqrt{1-\rho_{S\delta}^2}}\right)$, where $z=\dfrac{\ln s-\mu_S}{\sigma_S\sqrt{T-t}}$.

A 4. $E_t^Q\left[\delta_T 1_{\left\{\delta_T<d^*\right\}}\middle|S_T=s\right]=\eta_t\exp\left\{z\rho_{S\delta}\sigma_\delta\sqrt{T-t}\right\}\Phi_1\left\{\dfrac{d_2-\rho_{S\delta}\left(z-\rho_{S\delta}\sigma_\delta\sqrt{T-t}\right)}{\sqrt{1-\rho_{S\delta}^2}}\right\}$,

where $z=\dfrac{\ln s-\mu_S}{\sigma_S\sqrt{T-t}}$ and $\eta_t=\delta_t\exp\left\{\left(\sigma_D^2-\sigma_{VD}-\rho_{S\delta}^2\sigma_\delta^2/2\right)(T-t)\right\}$.

A 5. $\Phi_2(x,y;\rho)=\Phi_1(y)-\Phi_2(-x,y;-\rho)$ and $\Phi_2(x,y;\rho)=\Phi_1(x)-\Phi_2(x,-y;-\rho)$.

A 6. 1. $S_t \frac{\partial \Phi_2(a_1, a_2; \rho_{S\delta})}{\partial a_1} = Ke^{-r(T-t)} \frac{\partial \Phi_2(b_1, b_2; \rho_{S\delta})}{\partial b_1}$

2. $S_t e^{\sigma_{S\delta}(T-t)} \frac{\partial \Phi_2(c_1, c_2; -\rho_{S\delta})}{\partial c_1} = Ke^{-r(T-t)} \frac{\partial \Phi_2(d_1, d_2; -\rho_{S\delta})}{\partial d_1}$

3. $\frac{\partial a_1}{\partial S_t} = \frac{\partial b_1}{\partial S_t} = \frac{\partial c_1}{\partial S_t} = \frac{\partial d_1}{\partial S_t} = \frac{1}{S_t \sigma_S \sqrt{T-t}}$

Proof of Theorem 1.

Decompose $\Pi_C(t)$ as, $\Pi_C(t) = e^{-r(T-t)} E_t^Q[C_T] = \Pi_1 - \Pi_2 + \Pi_3 - \Pi_4$, where

$$\Pi_1 = e^{-r(T-t)} \cdot E_t^Q\left[S_T 1_{\{S_T \geq K\}} \cdot E_t^Q\left(1_{\{\delta_T \geq d^*\}} \middle| S_T\right)\right],$$

$$\Pi_2 = Ke^{-r(T-t)} \cdot E_t^Q\left[1_{\{S_T \geq K\}} \cdot E_t^Q\left(1_{\{\delta_T \geq d^*\}} \middle| S_T\right)\right],$$

$$\Pi_3 = (1-\alpha)e^{-r(T-t)} \cdot E_t^Q\left[S_T 1_{\{S_T \geq K\}} \cdot E_t^Q\left(\delta_T 1_{\{\delta_T < d^*\}} \middle| S_T\right)\right], \text{ and}$$

$$\Pi_4 = K(1-\alpha)e^{-r(T-t)} \cdot E_t^Q\left[1_{\{S_T \geq K\}} \cdot E_t^Q\left(\delta_T 1_{\{\delta_T < d^*\}} \middle| S_T\right)\right].$$

Let $Z_T = \frac{\ln S_T - \mu_S}{\sigma_S \sqrt{T-t}}$, a standardized normal random variable. By using A2-A 4, then

$$\Pi_1 = e^{-r(T-t)} E_t^Q\left[e^{Z_T \sigma_S \sqrt{T-t} + \mu_S} 1_{\{Z_T \geq -b_1\}} \Phi_1\left(\frac{b_2 + \rho_{S\delta} Z_T}{\sqrt{1-\rho_{S\delta}^2}} \right)\right] = S_t \Phi_2(a_1, a_2; \rho_{S\delta}),$$

$$\Pi_2 = Ke^{-r(T-t)} E_t^Q\left[1_{\{Z_T \geq -b_1\}} \cdot \Phi_1\left(\frac{b_2 + \rho_{S\delta} Z_T}{\sqrt{1-\rho_{S\delta}^2}} \right)\right] = Ke^{-r(T-t)} \Phi_2(b_1, b_2; \rho_{S\delta}),$$

$$\Pi_3 = (1-\alpha)\eta_t e^{-r(T-t)} E_t^Q\left\{ \exp\left[Z_T(\sigma_S + \rho_{S\delta}\sigma_\delta)\sqrt{T-t} + \mu_S \right] \cdot 1_{\{Z_T \geq -b_1\}} \Phi_1\left[\frac{d_2 - \rho_{S\delta}\left(Z_T - \rho_{S\delta}\sigma_\delta \sqrt{T-t}\right)}{\sqrt{1-\rho_{S\delta}^2}} \right]\right\}$$

$$= (1-\alpha) S_t \delta_t \exp\left\{ \left(\sigma_{S\delta} + \sigma_D^2 - \sigma_{VD}\right)(T-t) \right\} \Phi_2(c_1, c_2; -\rho_{S\delta}), \text{ and}$$

$$\Pi_4 = K(1-\alpha)\eta_t e^{-r(T-t)} E_t^Q \left\{ \exp\left(\rho_{S\delta}\sigma_\delta\sqrt{T-t}Z_T\right)\cdot 1_{\{Z_T \ge -b_1\}} \Phi_1\left(\frac{d_2 - \rho_{S\delta}\left(Z_T - \rho_{S\delta}\sigma_\delta\sqrt{T-t}\right)}{\sqrt{1-\rho_{S\delta}^2}} \right) \right\}$$

$$= K(1-\alpha)\delta_t \exp\left\{ \left(\sigma_D^2 - \sigma_{VD} - r\right)(T-t) \right\} \Phi_2\left(d_1, d_2; -\rho_{S\delta}\right).$$

Theorem 2 follows by applying A5 to Theorem 1. By partial differentiating upon the results of Theorem 1 with respect to S_t, together with A6, the result of Theorem 3 follows.

APPENDIX B

Proof of Theorem 4.

For convenience, the system of simultaneous equation (11) is re-numbered as follows:

$$\begin{cases} p_1 + p_2 + p_3 + p_4 = 1 & (b.1) \\ u_S d_S = 1 & (b.2) \\ u_\delta d_\delta = 1 & (b.3) \\ u_S(p_1+p_2) + d_S(p_3+p_4) = \exp\{r\Delta_n\} & (b.4) \\ u_\delta(p_1+p_4) + d_\delta(p_2+p_3) = \exp\left\{-\left(\sigma_V^2 - \sigma_D^2 - \sigma_\delta^2\right)\Delta_n/2\right\} & (b.5) \\ u_S^2(p_1+p_2) + d_S^2(p_3+p_4) = \exp\left\{\left(2r + \sigma_S^2\right)\Delta_n\right\} & (b.6) \\ u_\delta^2(p_1+p_4) + d_\delta^2(p_2+p_3) = \exp\left\{\left(\alpha_V + 3\alpha_D\right)\Delta_n\right\} & (b.7) \\ u_S u_\delta p_1 + u_S d_\delta p_2 + d_S d_\delta p_3 + d_S u_\delta p_4 = \exp\left\{\left(r + \alpha_D + \sigma_{S\delta\delta}\right)\Delta_n\right\} & (b.8) \end{cases}$$

Rearranging equations (b.1) leads to $p_1 + p_2 = 1 - (p_3 + p_4)$ and substituting this into $(b.4)$ results in

$$(u_S - d_S)(p_3 + p_4) = u_S - \exp(r\Delta_n). \qquad (b.9)$$

Result (b.9) leads equation $(b.6)$ to turn out $u_S^2 - \left\{ \exp(-r\Delta_n) + \exp\left[\left(r + \sigma_S^2\right)\Delta_n\right] \right\} u_S + 1 = 0$.

Thus $u_S = 1 + \sigma_S^2\Delta_n/2 + \sqrt{\sigma_S^2\Delta_n + \left(r^2 + r\sigma_S^2 + 3\sigma_S^4/4\right)\Delta_n^2} + o(\Delta_n) \approx \exp\left(\sigma_S\sqrt{\Delta_n}\right)$, and

$$d_S = 1 + \sigma_S^2\Delta_n/2 - \sqrt{\sigma_S^2\Delta_n + \left(r^2 + r\sigma_S^2 + 3\sigma_S^4/4\right)\Delta_n^2} + o(\Delta_n) \approx \exp\left(-\sigma_S\sqrt{\Delta_n}\right).$$

Similarly, results of equations (b.1) and (b.5), lead to

$$(u_\delta - d_\delta)(p_2 + p_3) = u_\delta - \exp\left\{-\left(\sigma_V^2 - \sigma_D^2 - \sigma_\delta^2\right)\Delta_n/2\right\}. \qquad (b.10)$$

Furthermore, combining (b.7) and (b.10), results in

$$u_\delta = 1 + \sigma_\delta^2\Delta_n/2 + \sqrt{\sigma_\delta^2\Delta_n + \left[\alpha_D\left(\sigma_\delta^2 + \alpha_D\right) + 3\sigma_\delta^4/4\right]\Delta_n^2} + o(\Delta_n) \approx \exp\left(\sigma_\delta\sqrt{\Delta_n}\right),$$

and $d_\delta = 1 + \sigma_\delta^2 \Delta_n / 2 - \sqrt{\sigma_\delta^2 \Delta_n + \left[\alpha_D\left(\sigma_\delta^2 + \alpha_D\right) + 3\sigma_\delta^4/4\right]\Delta_n^2} + o(\Delta_n) \approx \exp\left(-\sigma_\delta \sqrt{\Delta_n}\right).$

Finally, parameters $\{p_i\}$ could be solved via the equation, $Bx = \beta$, where

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ u_S u_\delta & u_S d_\delta & d_S d_\delta & d_S u_\delta \end{bmatrix}, \quad x^T = [p_1, p_2, p_3, p_4], \quad \beta^T = [1, \beta_1, \beta_2, \beta_3] \quad \text{and}$$

β_i's are determined from equations, (*b*.1), (*b*.9), (*b*.10) and (*b*.8). This completes the proof.

APPENDIX C

Sketch of the CBT algorithm

The time interval $[t, T]$ is divided into k_n subintervals of equal length Δ_n , with $\Delta_n = (T-t)/k_n$. Trading is supposed to occur at equidistant time points $t_{i,n} = t + i\Delta_n$, for $i = 0, 1, 2, \cdots, k_n$. The *j*-th node at time $t_{i,n}$ is referred to as the (i, j) node, and the stock price at the (i, j) node is

$$S_{i,j} = S_{0,0}(1+u_n)^j (1+d_n)^{i-j} = S_t \exp\{(2j-i)\sigma_S \sqrt{\Delta_n}\},$$

with $S_{0,0} = S_t$, for $0 \le j \le i \le k_n$. Here $u_n = \exp\left(\sigma_S \sqrt{\Delta_n}\right) - 1$, $d_n = \exp\left(-\sigma_S \sqrt{\Delta_n}\right) - 1$, $p_n = (r_n - d_n)/(u_n - d_n)$, $r_n = \exp(r\Delta_n) - 1$, $d_n < r_n < u_n$, and r is the risk-free interest rate.

Applying the double expectation property of bi-variate normal distribution, equation (5) is re-expressed as:

$$\Pi_C(t) = e^{-r(T-t)} E_t^Q \left\{ (S_T - K)^+ E_t^Q \left[1_{\{\delta_T \ge d^*\}} + (1-\alpha) \cdot 1_{\{\delta_T < d^*\}} \delta_T \middle| S_T \right] \right\}.$$

From statement (7), the conditional distribution of $\ln \delta_T$, given $\ln S_T = \ln S_{k_n, j}$, is normally distributed, say

$$\ln \delta_T \Big|_{S_T = S_{k_n, j}} \overset{d}{\sim} N_1\left(\alpha_{k_n, j}, \beta_{k_n}^2\right), \text{ for } 0 \le j \le k_n,$$

where $\alpha_{k_n, j} = \mu_{\delta, k_n} + \rho_{S\delta}\sigma_\delta \left(\ln S_{k_n, j} - \mu_{S, k_n}\right)/\sigma_S$, $\beta_{k_n}^2 = (T-t)\sigma_\delta^2\left(1 - \rho_{S\delta}^2\right)$,

$\mu_{\delta, k_n} = \ln \delta_t - (T-t)\left(\sigma_V^2 - \sigma_D^2\right)/2$, $\mu_{S, k_n} = \ln S_t + (T-t)\left(r - \sigma_S^2/2\right)$, and $\sigma_S > 0$.

Let $f_{i,j}$ be the arbitrage value of the vulnerable European call option at the (i, j) node, then

$$f_{i,j} = \exp(-r\Delta_n) \cdot \left[p_n f_{i+1, j+1} + (1-p_n) f_{i+1, j}\right], \text{ for } 0 \le j \le i < k_n,$$

with initial conditions $f_{k_n, j} = \xi_{k_n, j}$, for $0 \le j \le k_n$. Here

$$\xi_{k_n,j} = E_t^Q\left[C_T \mid S_T = S_{k_n,j}\right] = \left(S_{k_n,j} - K\right)^+\left[\psi_{k_n,j} + (1-\alpha)\pi_{k_n,j}\right],$$

where $\psi_{k_n,j} = E_t^Q\left[1_{\{\delta_T \ge d^*\}} \mid S_T = S_{k_n,j}\right] = \Phi_1\left(-g_{k_n,j}\right)$, $g_{k_n,j} = \left(\ln d^* - \alpha_{k_n,j}\right)/\beta_{k_n}$, and

$$\pi_{k_n,j} = E_t^Q\left[\delta_T 1_{\{\delta_T < d^*\}} \mid S_T = S_{k_n,j}\right] = \exp\left(\alpha_{k_n,j} + \beta_{k_n}^2/2\right)\cdot\Phi_1\left(g_{k_n,j} - \beta_{k_n}\right).$$

Beginning from initial values $\left\{f_{k_n,j}\right\}_{j=0}^{k_n}$, and moving backward throughout every node of the binomial lattice, the arbitrage price of the vulnerable European call option at the current time point, is determined to be $f_{0,0}$.