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Why a Negative Number Times a Negative Number Equals a Positive Number

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Abstract: One of the most mysterious mathematical topics taught in any elementary mathematics classroom is the concept that a negative time a negative equal a positive. This fundamental mathematical idea is listed in most elementary algebra text books as a rule without any justification for the validity of the rule. In this paper, I will present numerous mathematical arguments that attempt to justify this concept.
Key word: Negative and Positive Numbers;

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1. BRIEF HISTORY OF NEGATIVE NUMBERS

This section of the paper provides a brief historical account of negative numbers. According to numerous sources, among them Rajapakse (2012), who claimed that the concept of negative numbers was first introduced by the Indian mathematician Brahmagupta (c. 598–c. 670); and that Brahmagupta also devised the four rules of arithmetic (addition, subtraction, multiplication and division) using real numbers. In addition Brahmagupta introduced many fundamental concepts to basic mathematics, including the use of zero in the decimal number system, and the use of algebra to describe and predicting astronomical events.

There is disagreement among mathematics historians as to who first introduced the concept of negative number into mathematics. According to a listing in Wikipedia, Struik (1987, cited in Wikipedia):

> Negative numbers appear for the first time in history in the Nine Chapters on the Mathematical Art (Jiu zhang suan-shu), which in its present form dates from the period of the Han Dynasty (202 BC – AD 220).

According to Frey (2012) and others, Fibonacci and Cardano were both accredited with introducing the concept of negative numbers to Europe. Frey and others claimed that Fibonacci's book *Liber Abaci* contained problems with negative solutions, interpreted as debts (13^{th} century); and that Cardano's *Ars Magna* included negative solutions of equations and also had basic laws of operating with negative numbers (16^{th} century).

It took mathematicians for centuries to accept the notion of negative numbers, where the concept was regarded as "absurd" or "fictitious." Consider the paper by Rogers (1997), who asserted that:

> Although the first set of rules for dealing with negative numbers was stated in the 7th century by the Indian mathematician Brahmagupta, it is surprising that in 1758 the British mathematician Francis Maseres was claiming that negative numbers
>
> ... darken the very whole doctrines of the equations and make dark of the things which are in their nature excessively obvious and simple .

2. INTRODUCTION

This part of the paper is divided into two sections. The first section is entitled Graphical Method. In this section, several graphical techniques were used to show that a negative times a negative equals a positive, and for the second section entitled Algebraic Method several algebraic techniques was used to show that (-1)*(-1) =1.

3. GRAPHICAL METHOD -SECTION 1*A*

Probable, the simplest way to show that (-1)*(-1) =1, is to appeal to the method of the opposite of a real number. The method of opposite asserts that for a real number: if two numbers are the same distance from zero on the number line, then for real number *a*; -*a* is the additive inverse of *a* . To demonstrate this method, consider the following situations:

Number	Opposite
1	-1
-1	-1(-1)=1

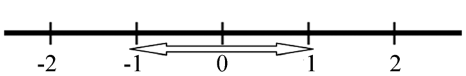


Figure 1
Real Number Line

Since -1 is a negative number, |-1|=-(-1)=1, given that the additive inverse of -1 is +1

It can also be seen from Figure 9, that -1 is opposite to +1, and that +1 is opposite to -1.

This demonstration shows that, the opposite of the negative of a number is that number.

Section 1*b*

Consider the following sequence of numbers: -16, -12, -8, -4, 0, 4, 8, 12, 16…..

This sequence of numbers can be rewritten in the form of a table, and on the number line as follows:

4(-4)=-16
3(-4)=-12
2(-4)=-8
1(-4)=-4
0(-4)=0
-1(-4)=4
-2(-4)=8
-3(-4)=12
-4(-4)=16

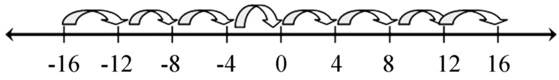


Figure 2
Real Number Line

This type of pattern recognition reasoning suggests that the product of two negative real numbers is a positive real number.

This result can be generalized by incorporate the concept of absolute values to describe multiplication as follows:

a) The product of two positive real numbers or two negative real numbers is the product of their absolute values.
b) The product of a positive and a negative number (order not important) is the opposite of the product of their absolute values.
c) The product of a zero and any real number is zero.

The following examples illustrate the concept of multiplication:

a) (-1)(-1)=|-1|·|-1|=1·1=1 ,
b) (1)(-1)=-(|1|·|-1|)=-(1·1)=-1 ,
c) (-1)(1)=-(|-1|·|1|)=-(1·1)=-1 ,
d) (-1)(0)=0 and (0)(-1)=0 .

Section 1*c*

The sequence of numbers from section 1*b*: -16, -12, -8, -4, 0, 4, 8, 12, 16… can be presented in tabular form as shown in Table 1, below:

Table 1
Sequence of Numbers in Tabular Form

-16	-12	-8	-4	0	4	8	12	16	…
(-4)(4)	(-3)(4)	(-2)(4)	(-1)(4)	(0)(4)	(1)(4)	(2)(4)	(3)(4)	(4)(4)	…..

The constant factor of 4 in Table 1, can be replaced with the real variable *k* (or *k* = 4) as shown in Table 2.

Table 2
The Coefficient of *k*

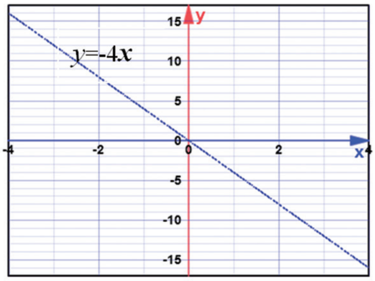
-4*k*	-3*k*	-2*k*	-1*k*	0*k*	1*k*	2*k*	3*k*	4*k*	….

The coefficient of *k* can be replaced with the real variable ***x***: -4, -3, -2, -1, 0, 1, 2, 3, 4,… as shown in Table 3.

Table 3
Rearrangements of Tables 1 and 2

x	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**
***k*=4**									
y*= - *kx	16	12	8	4	0	-4	-8	-12	-16

The constant multiplier of ***k*=4** represent the slope of the line in the ***x-y*** plane passing through the origin; the array of numbers can be represented by the formula ***y* = - *kx*** or ***y* = - 4*x***. A plot of this situation is given in Figure 3.



y=-4x

if x=-1

theny =-4x

y =-4(-1)=4

Figure 3
Graphical Representation

This result suggests that the product of two negative real numbers is a positive real number.

Section 2: Using algebraic method to show that (-1)*(-1) =1

Using the properties of the real numbers to show that **(-1)*(-1) =1.**

a) $\forall a \in \mathbb{R}, \exists$ an element -a such that a+(-a)=0, additive inverse.

b) Distributive property, $a(b+c)=ab+ac$.

Please see Appendix A and B, for an explanation of Ordered Fields and Equality.

Example 1

$$\text{If } (-1)(-1) = 1 \,. \tag{1}$$

Show that the left and right hand side of equation 1 is equal to zero.

Then	(-1)(-1) - 1 = 0 .	
Let -1= +(-1)(1)	(-1)(-1) + (-1)(1) = 0	Multiplicative Property
Factoring out (-1)	(-1)[(-1) + (1)] = 0	Distributive Property
Assume	(-1) + (1) =0	additive inverse property
It thus follows that	(-1)[0] = 0	
	0 = 0	Multiplicative property of zero
Result as required	0 = 0	Reflective property of equality

Example 2

Show that (-1)(-1) =1, using the distributive and the additive inverse property of the real numbers. Assume that the real number properties are true, for $a(b+c) = ab + ac$.

Given that $a = -1$, $b = 1$, $c = -1$ and $1 + (-1) = 0$.

Using the distributive property $a(b+c) = ab + ac$.

Substitution $$(-1)[1+(-1)] = (-1)(1) + (-1)(-1) \,. \tag{2}$$

Additive inverse on the left hand side $(-1)[0] = -1 + (-1)(-1)$.

$0 = -1 + (-1)(-1)$, adding 1 to both sides

Required result $1 = (-1)(-1)$.

Next consider the case where (-1)(-1) ≠1.

Proof by contradiction

Assume that a=-1, b=1, c=-1 and (-1)(-1) = -1.

Using distributive property: $a(b+c)=ab+ac$.

Substitution $$-1[\,1+(-1)] = (-1)(1)+(-1)(-1) \,. \tag{3}$$

Additive inverse $-1[0]= -1 -1$.

Result $0 = -2$, contradiction .

It there follows that $1 = (-1)(-1)$.

General Proof 1, show that $(-a)(-b) = ab$.

This proof relies on the distributional properties, and the additive inverse property of the real numbers.

Let a , b and x be any real numbers.

Consider the number x defined as follows: $x=ab + (-a)(b)+(-a)(-b)$. (4)

Factor out $-a$ from Equation(4) as follows: $x=ab + (-a)[(b) + (-b)]$.

Additive inverse $x=ab + (-a)[0]$, since $(b) + (-b) = 0$.

Thus $x=ab$.

Factor out b from Equation(4) as follows: $x=[a + (-a)](b) + (-a)(-b)$.

Additive inverse $x=[0](b) + (-a)(-b)$, since $a + (-a) = 0$.

Thus $x= (-a)(-b)$.

Since $ab = x$ and $x= (-a)(-b)$.

Transitivity of equality property $ab = (-a)(-b)$.

General Proof 2, show that (-1)(-*a*) =*a*

$(-a) + (-1)(-a) = 0= a+(-a)$ Axiom 5

$(-a)+ (-1)(-a) = (-a) + a$ Axiom 2

$a+ [(-a) + (-1)(-a)] = a + [(-a) + a]$ Axiom 1

$[a+ (-a)]+ [(-1)(-a)] = [a+(-a)] + a$ Axiom 3

$0 +(-1)(-a)=0+a$ Axiom 5

$(-1)(-a)+0=a+0$ Axiom 2

$(-1)(-a) = a$ Result as required Axiom 4

4. THE NATURAL OR COUNTING NUMBERS

The first type of number is the counting numbers:

The natural or counting numbers can be represented as follows: $N = \{1, 2, 3, 4, \ldots\}$

The natural numbers consist of the set of positive non-zero numbers or counting numbers. The set is denoted with the symbol, *N*. There is a first counting number, and for each counting number, there is a next counting number, or a successor. No counting number is its own successor. No counting number has more than one successor. No counting number is the successor of more than one other counting number. Only the number 1 is not the successor of any counting number (Peano axioms for the natural numbers-without proof).

The natural numbers are used to count physical objects in the real world.

The natural numbers system does not support division or negative numbers.

5. WHOLE NUMBERS

The second type of number is the whole numbers:

= {0,1, 2, 3, 4, . . . } the set is denoted with a symbol *W*. The whole numbers are the natural numbers together with zero. This number system allows us to add and multiply whole numbers. For example the sum of any two whole numbers is also a whole number: 4 + 20 = 24, and the product of any two whole numbers is a whole number ($4 \times 20 = 80$). This number system does not support subtraction and division..

6. THE INTEGERS

The third type of number system is the integers:

The integers are the set of numbers consisting of the natural numbers, their additive inverses and zero. This number system is usually symbolized as follows:

$$\square = \{..., -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5...\} .$$

Thus the sum, product, and difference of any two integers are also integer. However this is not true for division that is $1 \div 2$ is not an integer.

7. THE RATIONAL NUMBERS

The fourth type of number is the rationales:

The rational numbers are those numbers which can be expressed as a ratio between two integers.

$\because = \{ p/q \mid p, q \in Z$ and $q \neq 0 \}$ or equivalently $\because = Z \, x \, (Z \backslash \{0\})$.

Where ***p*** and ***q*** are integers; the set of fractions (pairs of integers) with non-zero denominator.

with the following relation $\because$ **:**$p_1/q_1 \,\square\, p_2/q_2$ if and only $p_1q_2 = p_2q_1$

The set of rational numbers are ratios of integers (where the divisor is non-zero), such as 2/5, -2/11, etc.. Note that the integers are included among the rational numbers, for example, the integer 3 can be written as 3/1, or even 8/2 . Additional examples of rational numbers are 1/2 -4/7, 6, and 0. Any rational number may be (written as a terminating number, like 0.4 or a repeating decimal number, like 0.3333…). A rational number is a number that can be used to do mathematics: that is calculations, solve equations that do not involve radicals, and used to represent measurements. In general arithmetic operations that involve the sum, product, quotient, and the difference of any two integers are also a rational. However this is not true for irrational, that is, $\sqrt{2}$ is not a rational number.

8. THE IRRATIONAL NUMBER

The fifth type of number is the irrationals:

$L = \{ x \mid$ is a real number, but x cannot be written as a quotient of integers$\}$

An irrational number is a number that cannot be written as a ratio (or fraction). In decimal form, this number system never ends or repeats. The Pythagorean discovered that not all numbers are rational; there are equations that cannot be solved using ratios of integers.

It turns out that the rational numbers are not enough to describe the world. Consider, for example, a right triangle whose two short sides are each 1 unit long. Then by the Pythagorean Theorem the longest side of the triangle, the hypotenuse, has a length whose square equals 2. This length is usually referred to as the square root of $2(\sqrt{2})$.

Prove that: $\sqrt{2}+\sqrt{3}$ is an irrational number. We will prove the statement, by appealing to the technique of proof by contradiction. Assume that p and q are integers and $p \neq 0$ with no common integer factors, then $\sqrt{2}+\sqrt{3}=\frac{p}{q}$, assume that this fraction is in its simplest form.

Rearrange: $\sqrt{3}=\frac{p}{q}-\sqrt{2}$ - squaring both sides:

$$3=\frac{p^2}{q^2}+2-2\sqrt{2}\,\frac{p}{q} \quad \text{Rearrange} \quad 2\sqrt{2}\,\frac{p}{q}=\frac{p^2}{q^2}-1$$

$$2\sqrt{2}\frac{p}{q}=\frac{p^2}{q^2}-1 \quad \Rightarrow 2\sqrt{2}\,\frac{p}{q}=\frac{p^2-q^2}{q^2} \Rightarrow \sqrt{2}=\frac{p^2-q^2}{2pq}$$

$\frac{p^2-q^2}{2pq}$ is a rational number, because p and q are integers, this implies that $\sqrt{2}$ is a rational number, which is impossible, and so $\sqrt{2}+\sqrt{3}$ is an irrational number.

Thus the sum, product, quotient, rational and the difference of any two irrational is also an irrational number. However this is not true for complex number, that is, numbers of the form, $a+bi$ is not an irrational number.

The union of the natural, whole, integers, rational and irrational is usually referred to as the real numbers, symbolized as follows: □=⋰∪□∪□∪∵∪ We will have much more to say about the real numbers in late sections

9. THE COMPLEX NUMBERS

The sixth type of number is complex numbers:

The study of negative numbers can be expanded to include complex numbers or the so-called imaginary numbers.

The expansion of the number system comes in succession. The complex numbers come last.

Negative number was needed to solve equations of the form $a+x=b$, where $a>b$.

The introduction of the real number system was in response to the need to solve equation of the form $x^2=2$.

The introduction of the complex number system was in response to the need to solve equation of the form $x^2+1=0$.

In mathematics, a complex number is a number of the form $a+bi$,where $\boldsymbol{a}$ and $\boldsymbol{b}$ are real numbers, and $\boldsymbol{i}$ is the imaginary unit, with the property $i^2=-1$ The real number $\boldsymbol{a}$ is called the real part of the complex number, and the real number $\boldsymbol{b}$ is the imaginary part. Real numbers may be considered to be complex numbers with an imaginary part of zero; that is, the real number a is equivalent to the complex number $\boldsymbol{a}+0\boldsymbol{i}$. The basic number system can be characterized as followed: ⋰⊂□⊂□⊂∵⊂□⊂⋯⊂∐.

10. AN ORDERED SET

The real numbers have the property that they are ordered, which means that given any two different numbers we can say that one number is greater or less than the other number.

A formal way of saying this is: Symbols are used to show how the size of one number compares to another. These symbols are < (less than), > (greater than), and = (equals.)

For any two real numbers a and b, one and only one of the following three statements is true:

a) a is less than b, (expressed as $a < b$) ,
b) a is equal to b, (expressed as $a = b$),
c) a is greater than b, (expressed as $a > b$).
The field $\mathbb{C}$ of complex numbers is not an ordered field under any ordering.

Proof

Suppose $i>0$. Squaring both sides gives $i^2>0$ or $-1>0$: Adding 1 to both sides of the inequality gives $0>1$; and so we have a contradiction.

According to Bogomolny (2015) Square roots of negative numbers appeared in Ars Magna (1545) by Girolamo Cardano, who would consider several forms of quadratic equations (e.g., $x^2 + px = q$, $px - x^2= q$, $x^2 = px + q$) just in order to avoid using negative numbers. Which is hardly surprising in view of the fact that the tools Cardano used are usually described asgeometric algebra. This is yet in the tradition of, say, Euclid II.5 and II.6, Al-Khowarizmi (Smith, pp.446-447)], and many others. Algebraic symbolism was still evolving and cumbersome and the proofs have been geometric. Cardano's internal conflict is tangible in his writing. He handles the problem (Source Book, pp.201-202) that nowadays would be described as solving the quadratic equation $x^2 - 10x + 40 = 0$.

CONCLUSION

The main goal of this paper is to introduce the readers to techniques and ideas associated with the notion of (-1)(-1) =1. The concepts are introduced in a concrete and elementary way to allow for a wide readership. It is my fervent desire for anyone from a motivated high school student interested in mathematics to college students specializing in mathematics, to find this topic sufficiently intriguing that they will want to carry out further research on this topic.

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