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Asymptotic Regularity and Exponential Attractors for Nonclassical Diffusion Equations With Critical Exponent

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Abstract: In this paper, we consider the dynamical behavior of the nonclassical diffusion equation when nonlinearity is critical for both two cases: the forcing term belongs to $H^{-1}(\Omega)$ and $L^2(\Omega)$. For the case the forcing term only belongs to $H^{-1}(\Omega)$, based on the asymptotic regularity in *Dynamical Systems: An International Journal*, *26*(4), (2011), 391–400, we prove the existence of exponential attractors in weak topological space $H_0^1(\Omega)$. For the case the forcing term belongs to $L^2(\Omega)$, we prove the asymptotic regularity of the solutions and exponential attractors.

Key words: Nonclassical diffusion equations; Exponential attractors; Absorbing set; Asymptotic regularity; Critical exponent.

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1. INTRODUCTION

We consider the longtime behavior for the following nonclassical equation

$$u_t - \Delta u_t - \Delta u + f(u) = g(x), \quad \text{in} \quad \Omega \times \mathbb{R}_+, \tag{1.1}$$
$$u = 0, \quad \text{on} \quad \partial\Omega, \tag{1.2}$$
$$u(x, 0) = u_0, \quad x \in \Omega, \tag{1.3}$$

where Ω is an open bounded set of $\mathbb{R}^n (n \geq 3)$ with smooth boundary $\partial\Omega$, the nonlinearity satisfies:

(F_1) $\lim\inf_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1$,

(F_2) $\mid f'(s) \mid \leq C(1+ \mid u \mid^{\frac{4}{n-2}})$, $s \in \mathbb{R}$,

where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, and we also assume $f \in \mathcal{C}^1(\mathbb{R})$ with $f(0) = 0$.

In 1980, Aifantis in [1] pointed out that the classical reaction-diffusion equation does not contain each aspect of the reaction-diffusion problem, and it neglects viscidity, elasticity and pressure of medium in the process of solid diffusion and so on. In the sequel, Aifantis found out that the energy constitutional equation revealing the diffusion process is different along with the different property of the diffusion solid. Therefore, he constructed the mathematic model by some concrete examples, which contains viscidity, elasticity and pressure of medium, that is the nonclassical parabolic equation. The term $-\Delta u_t$ denote the pressure, viscoelasticity and memory, e.g. see [1-5].

The longtime behavior of Equation (1.1) has been extensively studied by several authors in [6-17,21,22] and references therein. In [12] the existence of a global attractor for the autonomous case has been shown provided that the nonlinearity is critical and $g(x) \in H^{-1}(\Omega)$; In [15] the asymptotic regularity for the autonomous case has been shown provided that the nonlinearity is critical and $g(x) \in H^{-1}(\Omega)$; Furthermore, for the non-autonomous, the existence of a uniform attractor, asymptotic regularity and exponential attractors have been scrutinized when the time-dependent forcing term $g(x, t)$ only satisfies the translation bounded instead of translation compact, namely, $g(x, t) \in L_b^2(\mathbb{R}, L^2(\Omega))$. A similar problem was discussed in [13] by virtue of the standard method based on the so-called squeezing property. The dynamics of (1.1) acted on a unbounded domain $\mathbb{R}^N$ has been considered in [14].

To our best knowledge, the exponential attractors when $g \in H^{-1}$, the asymptotic regularity and exponential attractors when $g \in L^2$ for $(1.1) - (1.3)$ remain open. Since Equation (1.1) contains the term $-\Delta u_t$, it is different from the usual reaction-diffusion equation essentially. For example, the reaction diffusion equation has some smoothing effect, e.g., although the initial data only belongs to a weaker topology space, the solution will belong to a stronger topology space with higher regularity. However, for Equation (1.1), if the initial data u_0 belongs to $H_0^1(\Omega)$, then the solution $u(t, x)$ with $u(0, x) = u_0$ is always in $H_0^1(\Omega)$ and has no higher regularity because of $-\Delta u_t$. Meanwhile the nonlinearity with critical exponent also brings some difficulties.

This paper is organized as following: In Section 2, we recall some basic definitions and related theorems that will be used later. In Section 3, we prove the existence of exponential attractor in $H_0^1(\Omega)$ when $g(x) \in H^{-1}(\Omega)$. In Section 4, we prove the asymptotic regularity when $g(x) \in L^2(\Omega)$. In Section 5, we prove the exponential attractors in $H_0^1(\Omega)$ when $g(x) \in L^2(\Omega)$.

2. PRELIMINARIES

In this section, we iterate some notations and abstract results.

Let $A = -\Delta$ with the domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, we need the continuous embedding

$$\mathcal{D}(A^{\frac{s}{2}}) \hookrightarrow L^{2n/(n-2s)}(\Omega) \quad s \in [0, \frac{n}{2}). \tag{2.1}$$

and the interpolation results: given $s > r > q$, for any $\varepsilon > 0$ there exists $C_\varepsilon = C_\varepsilon(s, r, q)$ such that

$$\| A^{r/2}u \| \leq \varepsilon \| A^{s/2}u \| + C_\varepsilon \| A^{q/2}u \| \quad \forall u \in D(A^{s/2}). \tag{2.2}$$

We set $\mathcal{H}_s = \mathcal{D}(A^{\frac{s}{2}})$ for $0 \leq s \leq 2$, then $\mathcal{H}_0 = L^2(\Omega)$, $\mathcal{H}_1 = H_0^1(\Omega)$, $\mathcal{H}_2 = H_0^1(\Omega) \cap H^2(\Omega)$.

We recall some results in [7,9,15,16] about the exponential attractors:

Definition 2.1. *A compact set $\mathscr{M} \subseteq \mathscr{B}$ is called an exponential attractor for $(S(t), \mathscr{B})$ if:*

1) $\mathscr{M}$ has finite fractal dimension;

2) $\mathscr{M}$ is a positive invariant set of $S(t)$: $S(t)\mathscr{M} \subseteq \mathscr{M}$, for all $t > 0$;

3) $\mathscr{M}$ is an exponentially attracting set for the semigroup $\{S(t)\}_{t\geq 0}$; i.e. there exist universal constants α,$\beta > 0$ such that

$$\text{dist}_X(S(t)u, \mathscr{M}) \leq \alpha e^{-\beta t}, \quad \forall u \in \mathscr{B},\ t > 0,$$

where dist denotes the nonsymmetric Hausdorff distance between sets.

We shall use the following abstract criterions to establish the existence of exponential attractor (see [13,15,16] for detail).

Lemma 2.1. *Let E and E_1 be two Banach spaces such that E_1 is compactly embedded into E and let B be a bounded subset of E_1. Assume that there exists a time $t_* > 0$ such that the following hold:*

(i) the map $(t, u_0) \longmapsto S(t)u_0 : [0, t_*] \times \mathbb{B} \longrightarrow \mathbb{B}$ is Lipschitz continuous (with the metric inherited from E);

(ii) S admits a decomposition of the form

$$S = S_0 + S_1, \quad S_0 : \mathbb{B} \to E,\ S_1 : \mathbb{B} \to E_1,$$

where S_0 and S_1 satisfy the conditions

$$\|S_0(u_1) - S_0(u_2)\|_E \leq \frac{1}{8}\|u_1 - u_2\|_E, \quad \forall\ u_1,\ u_2 \in \mathbb{B},$$

and

$$\|S_1(u_1) - S_1(u_2)\|_{E_1} \leq K\|u_1 - u_2\|_E, \quad \forall\ u_1,\ u_2 \in \mathbb{B}.$$

3. EXPONENTIAL ATTRACTORS IN $H_0^1(\Omega)$ WHEN $g(x) \in H^{-1}(\Omega)$

In order to construct an exponential attractor in $H_0^1(\Omega)$ when $g(x) \in H^{-1}(\Omega)$, we list some results introduced in [15]:

Lemma 3.1. [15] *Let f satisfy conditions $(F_1)-(F_2)$ and $g \in H^{-1}(\Omega)$. Then for any $u_0 \in \mathcal{H}$ and any $T > 0$, there is a unique solution u of $(1.1)-(1.3)$ such that*

$$u \in C([0,T];\mathcal{H}) \cap L^\infty([0,\infty);\mathcal{H}).$$

Moreover, the solution continuously depends on the initial data in $\mathcal{H}$.

According to Lemma 3.1 above, we can define the solution semigroup, that is

$$S(t) : \mathcal{H} \to \mathcal{H},$$

$$u_0 \to u(t) = S(t)u_0, \quad \forall u_0 \in \mathcal{H}, \quad t \geq 0.$$

Lemma 3.2. [15] *Let $f \in \mathcal{C}^1(\mathbb{R})$ and satisfy $(F_1)-(F_2)$, $g \in H^{-1}(\Omega)$ and $\{S(t)\}_{t\geq 0}$ be the semigroup generated by the weak solution of $(1.1)-(1.3)$ in the natural energy space $H_0^1(\Omega)$ Then, for any $\nu \in [0, \frac{N-1}{2})$, there exist a subset B_ν, positive constant μ and a monotone increasing function $Q_\nu(\cdot)$ such that for any bounded set $B \subset H_0^1(\Omega)$,*

$$dist_{H_0^1(\Omega)}(S(t), B_\nu) \leq Q_\nu(\| B \|_{H_0^1(\Omega)})e^{-\mu t}, for\ all\ t \geq 0,$$

where B_ν and $Q_\nu(\cdot)$ depend on ν but μ is independent of ν; B_ν satisfying

$$B_\nu = \{z \in H_0^1(\Omega) : \| u - \phi(x) \|_{H^{1+\nu}} \leq \Lambda_\nu < \infty\}$$

for some positive constant Λ_ν; and $\phi(x)$ is the unique solution of the following elliptic equation

$$\begin{cases} \Delta\phi + f(\phi) + (l+1) = g(x) - g^\eta, \in \Omega \\ \phi|_{\partial\Omega} = 0, \end{cases}$$

where the constant $l(>0)$, $g^\eta \in L^2(\Omega)$ such that $\|g - g^\eta\|_{H^{-1}} < \eta \ll 1$.

Lemma 3.3. [15] *Under the assumption of lemma 3.1, for any bounded (in H^ν) subset $B_1 \subset H^\nu$, if the initial data $u_0 \in \phi(x) + B_1$, then the solution $u(t)$ of $(1.1)-(1.3)$ also satisfies a similar estimate, more precisely, we have*

$$\| S(t)u_0 - \phi(x) \|_{H^\nu}^2 = \| u(t) - \phi(x) \|_{H^\nu}^2 \leq M_{B_1}, \quad \forall \quad t \geq 0.$$

where the constant M_{B_1} depends only on B and the H^ν-bound of B_1.

Based on the asymptotic regularity obtained above, we want to construct the exponential attractors in $H_0^1(\Omega)$. For this, we need to assume further that the nonlinearity $f(\cdot)$ satisfies the following technical conditions, they are very natural if our nonlinearity is a polynomial of order 5(see [18] for detail).

For each fixed $s \in \mathbb{R}$, $f(\cdot)$ admits a decomposition $f(\cdot) = f_1(\cdot) + f_2(\cdot)$, where $f_i(\cdot) \in \mathcal{C}^1(\mathbb{R})$ and satisfies: for any $r \in \mathbb{R}$,

$$\left| \frac{\partial}{\partial r} f_2(r+s) \right| \leq C(1+ | r |^4 + | s |^\gamma), \quad 0 \leq \gamma < 4, \tag{3.1}$$

$$\left| \frac{\partial}{\partial r} f_1(r+s) \right| \leq C | s |^4 . \tag{3.2}$$

Now, we begin to construct the exponential attractors of $\{S(t)\}_{t\geq 0}$ in $H_0^1(\Omega)$. For each fixed $\nu \in [0, \frac{N-1}{2})$, we denote

$$\mathbf{B}_\nu = \overline{\bigcup_{t\geq 1} S(t)\mathcal{B}_\nu}$$

where $\mathcal{B}_\nu$ is the set obtained in lemma 3.2. Then according to lemma 3.3, we get that

$$\| \mathbf{B}_\nu - \phi(x) \|_{H^\nu} < \infty. \tag{3.3}$$

For $\varepsilon \in (0, 1)$ to be determined later, we decompose $\phi(x) = \phi_0(x) + \phi$ with

$$\phi_0(x) \in H^2(\Omega) \cap H_0^1(\Omega) \ \ and \ \ \| \nabla\phi_\varepsilon(x) \| \leq \varepsilon. \tag{3.4}$$

For each initial data $u_0 \in \mathbf{B}_\nu$, decomposing the solution $u(t)$ of (1.1) – (1.3) as

$$u(t) = S_1(t)u_0 + S_2(t)u_0.$$

where $v = S_1(t)u_0$ and $w = S_2(t)u_0$ solves the following equations respectively:

$$\begin{cases} v_t - \Delta v_t - \Delta v + f_1(u) = g(x), \\ v(0) = u_0, \\ v \mid_{\partial\Omega} = 0, \end{cases} \tag{3.5}$$

and

$$\begin{cases} w_t - \Delta w_t - \Delta w + f_2(u) = 0, \\ w(0) = 0, \\ w \mid_{\partial\Omega} = 0. \end{cases} \tag{3.6}$$

Lemma 3.4. *Let f satisfy conditions $(F_1) - (F_2)$, (3.1) – (3.2) and $g \in H^{-1}(\Omega)$. Then following two estimates hold:*

$$\|S_1(t_*)u_1 - S_1(t_*)u_2\|_{\mathcal{H}} \leq \frac{1}{8}\|u_1 - u_2\|_{\mathcal{H}}, \quad \forall\ v_1,\ v_2 \in \mathbf{B}_\nu,$$

and

$$\|S_2(t_*)u_1 - S_2(t_*)u_2\|_{\mathcal{H}^\delta} \leq K\|u_1 - u_2\|_{\mathcal{H}}, \quad \forall\ v_1,\ v_2 \in \mathbf{B}_\nu,$$

where $\delta = \min\{\nu, \frac{n+2-(n-2)\gamma}{2}\}$ and γ is given in (3.1).

Note that for the nonlinearity $f_i(u)$, we would take $r = u - \phi_\varepsilon$ and $s = \phi_\varepsilon$ (fixed) corresponding to the above assumptions (3.1) – (3.2). According to (3.3) – (3.4), we get that

$$\| u - \phi(x) \|_{H^{1+\nu}} \leq M < \infty \ \ \forall t \geq 0, \ \ u_0 \in \mathbf{B}_\nu.$$

Hence, by taking first t_* large enough, then taking ε (in(3.2)) small enough. We can deduce the following estimates by some routine difference calculations(e.g., see[18,12]), so we omit it here.

Lemma 3.5. *Let the assumptions of lemma 3.4 hold, then for every $T > 0$, the mapping $(t, u) \mapsto S(t)u$ is Lipschitz continuous on $[0, T] \times \mathbf{B}_\nu$.*

Proof : For $u_1,\, u_2 \in \mathbf{B}_\nu$ and $t_1,\, t_2 \in [0,T]$ we have

$$\| S(t_1)u_1 - S(t_2)u_2 \|_{\mathcal{H}} \leq \| S(t_1)u_1 - S(t_1)u_2 \|_{\mathcal{H}} + \| S(t_1)u_2 - S(t_2)u_2 \|_{\mathcal{H}} .$$

The first term of the above inequality is handled by Lemma 1 (the solution continuously depends on the initial data in $\mathcal{H}$). Concerning the second one, because of the estimates about $\| u_t(y) \|_{\mathcal{H}}\, l$(in [8,10,13]) , we obtain

$$\| u(t_1) - u(t_2) \|_{\mathcal{H}} \leq \left| \int_{t_1}^{t_2} \| u_t(y) \|_{\mathcal{H}}\, dy \right| \leq L \mid t_1 - t_2 \mid .$$

Hence

$$\| S(t_1)u_1 - S(t_2)u_2 \|_{\mathcal{H}} \leq L[\mid t_1 - t_2 \mid + \| u_1 - u_2 \|_{\mathcal{H}}],$$

for some $L = L(T) \geq 0$.

Theorem 3.6. *(Exponential attractors in $H_0^1(\Omega)$ when $g(x) \in H^{-1}(\Omega)$) Let Ω is a bounded open set of $\mathbb{R}^n (n \geq 3)$ with smooth boundary $\partial\Omega$, and f satisfy conditions $(F_1) - (F_2)$, (3.1) − (3.2) and $g \in H^{-1}(\Omega)$. Then the semigroup of operators $\{S(t)\}_{t\geq 0}$ generated by (1.1) − (1.3) has an exponential attractor in $H_0^1(\Omega)$.*

Remark 3.7 : When the forcing term g only belong to $H^{-1}(\Omega)$, the attractors has no higher regularity than $H_0^1(\Omega)$.

4. ASYMPTOTIC REGULARITY WHEN $g(x) \in L^2(\Omega)$

Similar to the case $g \in H^{-1}(\Omega)$, we can also get the following results, we just list them in below.

Lemma 4.1. *Let f satisfy conditions $(F_1) - (F_2)$ and $g(x) \in L^2(\Omega)$. Then for any $u_0 \in \mathcal{H}$ and any $T > 0$, there is a unique solution u of (1.1) − (1.3) such that*

$$u \in C([0,T]; \mathcal{H}) \cap L^\infty([0,\infty); \mathcal{H}).$$

Moreover, the solution continuously depends on the initial data in $\mathcal{H}$.

According to Lemma 4.1 above, we can define the solution semigroup, that is

$$S(t) : \mathcal{H} \to \mathcal{H},$$
$$u_0 \to u(t) = S(t)u_0, \quad \forall u_0 \in \mathcal{H}, \quad t \geq 0.$$

Lemma 4.2. *Under the assumptions of lemma 4.1. Then there exists a positive constant N_0, such that for any bounded subset B in $\mathcal{H}_1$, there is a $t_0 = t_0(\| B \|_{\mathcal{H}_1})$ such that*

$$\| u(t) \|_{\mathcal{H}_1}^2 \leq M_0^2, \quad for \quad t \geq t_0 = t_0(\| B \|_{\mathcal{H}_1}). \tag{4.1}$$

Lemma 4.3. *(Bounded absorbing set) Under the assumptions of lemma 4.1, then for any bounded subset $B \subset \mathcal{H}_1$, there exists a constant M_0, there is a $t_0 = t_0(\| B \|_{\mathcal{H}_1})$ such that*

$$\| S(t)u_0 \|_{\mathcal{H}_1}^2 \leq M_0^2, \quad for \quad t \geq t \geq t_0 \ \ and \ \ z_0 \in B. \tag{4.2}$$

The proof of lemma 4.2 and lemma 4.3 are similar to the case $g \in H^{-1}(\Omega)$, we omit it here.

For the nonlinearity f , following [12, 15, 18, 19] we decompose f as follows:

$$f = f_0 + f_1,$$

where $f_0, f_1 \in C(\mathbb{R})$ satisfy

$$| f_0(u) |\leq c(| u | + | u |^{(n+2)/(n-2)}), \quad \forall u \in \mathbb{R}, \tag{4.3}$$

$$f_0(u)u \leq 0, \quad \forall u \in \mathbb{R}, \tag{4.4}$$

$$| f_1(u) |\leq c(1+ | u |^{\gamma}), \quad \forall u \in \mathbb{R} \quad \text{and} \quad 0 < \gamma < \frac{n+2}{n-2}, \tag{4.5}$$

$$\lim\inf f_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1, \tag{4.6}$$

$$\sigma = \min\{\frac{1}{4}, \frac{n+2-\gamma(n-2)}{2}\}, \tag{4.7}$$

where γ is defined by (4.7).

In order to obtain the regularity estimates later, we also decompose the solution into the sum

$$S(t)u(t) = D(t)u_0 + K_g(t)u_0,$$

where $v = D(t)u_0$ and $w = K_g(t)u_0$ solves the following equations respectively:

$$\begin{cases} v_t - \Delta v_t - \Delta v + f_0(u) = 0, \\ v(0) = u_0, \\ v |_{\partial\Omega}= 0, \end{cases} \tag{4.8}$$

and

$$\begin{cases} w_t - \Delta w_t - \Delta w + f(u) - f_2(u) = g(x), \\ w(0) = 0, \\ w |_{\partial\Omega}= 0. \end{cases} \tag{4.9}$$

Lemma 4.4. *Let f satisfy conditions $(F_1) - (F_2)$ and $g(x) \in L^2(\Omega)$. Then there exists a positive constant k_0 such that for every $t \geq t_1$,*

$$\| D(t)u_0 \|^2_{\mathcal{H}_1}\leq Q_1(\| u_0 \|_{\mathcal{H}_1})\mathrm{e}^{-k_0 t}, \tag{4.10}$$

where $Q_1(\cdot)$ is an increasing function on $[0, \infty)$.

Lemma 4.5. *Let f satisfy conditions $(F_1) - (F_2)$ and $g(x) \in L^2(\Omega)$. Then there exists a positive constant M such that for every $t \geq t_2$,*

$$\| K_g(t)u_0 \|^2_{\mathcal{H}_{1+\sigma}}\leq M, \tag{4.11}$$

where σ is given in (4.7).

The proof of lemma 4.4 and lemma 4.5 are similar to the proof of Lemma 3.3 and Lemma 3.4 in [12] for the case $g \in H^{-1}(\Omega)$, we omit it here.

Now, similar to that in [20] (see [12,15] also), based on Lemmas 4.4 and 4.5 above, we can decompose $u(t)$ as follows.

Lemma 4.6. *Under the assumptions of lemma 4.1, and $u(t)$ be the solution of equations $(1.1)-(1.3)$ with the initial data $u_0 \in \mathcal{H}_1$. Then for any $\epsilon > 0$, there are positive constants C_ϵ and K_ϵ, such that*

$$u(t) = v_1(t) + w_1(t), \quad for \quad all \quad t \geq 0, \tag{4.12}$$

where $v_1(t)$, $w_1(t)$ satisfy estimates as follows:

$$\| A^{\frac{1+\sigma}{2}} w_1(t) \|^2 \leq K_\epsilon, \quad t \geq 0 \tag{4.13}$$

and for every $t \geq s \geq 0$,

$$\int_s^t \| \nabla v_1(r) \|^2 \, \mathrm{d}r \leq \epsilon(t-s) + C_\epsilon, \tag{4.14}$$

where the constants C_ϵ and K_ϵ depending on ϵ, σ.

In what follows, we begin to establish the asymptotic regularity of the solutions. Noting that with minors change the proof from Lemma 4.7 to Theorem 4.11 in [12], we can get the following lemmas and Theorem, so we just list them in below.

Lemma 4.7. *Let f satisfy conditions $(F_1)-(F_2)$ and $g(x) \in L^2(\Omega)$. For any bounded (in $\mathcal{H}_1$) set $B \subset \mathcal{H}_1$, there exists a positive constant $J_{\|B\|_{\mathcal{H}_1}}$ which depends only on the $\mathcal{H}_1$-bounds of B, such that*

$$\| K_g(t)u_0 \|^2_{\mathcal{H}_{1+\sigma}} \leq J_{\|B\|_{\mathcal{H}_1}}, \quad for \quad all \quad t \geq 0 \quad and \quad u_0 \in B, \tag{4.15}$$

where σ is given in (4.7).

Lemma 4.8. *Let f satisfy conditions $(F_1)-(F_2)$ and $g(x) \in L^2(\Omega)$. Assume B_σ is an arbitrary bounded set in $\mathcal{H}_{1+\sigma}$. Then there exists a constant $M_{\|\mathcal{H}_{1+\sigma}\|_{B_{1+\sigma}}}(>0)$ such that*

$$\| S(t)(t)u_0 \|^2_{\mathcal{H}_{1+\sigma}} \leq M_{\|B_{1+\sigma}\|_{\mathcal{H}_{1+\sigma}}}, \quad for \quad all \quad t \geq 0 \quad and \quad u_0 \in B_\sigma, \tag{4.16}$$

where σ is given in (4.7).

Based on Lemma 4.7 and 4.8 above, we can perform a bootstrap argument to get the asymptotic regularity.

Lemma 4.9. *Let f satisfy conditions $(F_1)-(F_2)$ and $g(x) \in L^2(\Omega)$ and $\sigma \leq \theta \leq 1$. Then for any bounded $B_\theta \subset \mathcal{H}_{1+\theta}$. Then there exists a constant $M_{\|B_\theta\|_{\mathcal{H}_{1+\sigma}}}(>0)$ such that*

$$\| S(t)(t)u_0 \|^2_{\mathcal{H}_{1+\theta}} \leq M_{\|B_\theta\|_{\mathcal{H}_{1+\theta}}}, \quad for \quad all \quad t \geq 0 \quad and \quad u_0 \in B_\theta. \tag{4.17}$$

Lemma 4.10. *Let f satisfy conditions $(F_1)-(F_2)$ and $g(x) \in L^2(\Omega)$ and $\theta \in [\sigma, 1-\min\{\sigma, \frac{4\sigma}{n-2}\}]$, and assume that the initial data set B_θ is bounded in $\mathcal{H}_{1+\theta}$, then the decomposed ingredient w(t) (the solutions of (4.9)) satisfies that*

$$\| K_g(t)u_0 \|^2_{\mathcal{H}_{1+\theta+s_0}} \leq J_{\|B\|_{\mathcal{H}_{1+\theta}}}, \quad for \quad all \quad t \geq 0 \quad and \quad u_0 \in B_\theta, \tag{4.18}$$

where $s_0 = \min\{\sigma, \frac{4\sigma}{n-2}\}$.

Theorem 4.11. *(Asymptotic regularity when $g(x) \in L^2(\Omega)$) Let Ω is a bounded open set of $\mathbb{R}^n (n \geq 3)$ with smooth boundary $\partial\Omega$, and f satisfy conditions $(F_1)-(F_2)$, (3.1)−(3.2) and $g \in H^{-1}(\Omega)$. Then there exist a bounded (in $\mathcal{H}_2$) set $\mathcal{B}_1 \subset \mathcal{H}_2$, a positive constant λ and a monotonically increasing function $Q(\cdot)$ such that: for any bounded (in $\mathcal{H}_1$) set $B \subset \mathcal{H}_1$ and $t \geq 0$, the following estimate holds:*

$$dist_{\mathcal{H}_1}(S(t)B, \mathcal{B}_1) \leq Q(\| B \|_{\mathcal{H}_1}) e^{-\lambda t},$$

where $dist_{\mathcal{H}_1}$ is the usual Hausdorff semidistance in $\mathcal{H}_1$.

Remark 4.12 : When the forcing term g belong to $L^2(\Omega)$, the attractors has no higher regularity than $\mathcal{H}^2(\Omega)$.

5. EXPONENTIAL ATTRACTORS IN $H_0^1(\Omega)$ WHEN $g(x) \in L^2(\Omega)$

Similar to the proof of Lemma 5.5 in [12], we get that

Lemma 5.1. *Let f satisfy conditions $(F_1)-(F_2)$ and $g(x) \in L^2(\Omega)$, there exists a positive M, such that for any bounded (in $\mathcal{H}_2$) subset $B \subset \mathcal{H}_2$, there is a $T > 0$ such that*

$$\| S(t)u_0 \|_{\mathcal{H}_2}^2 \leq M_2, \quad for \quad all \quad t \geq T \quad and \quad u_0 \in B. \tag{5.1}$$

In what follows, we set

$$\mathbb{B} = \{u \in \mathcal{H}_2 : \| \Delta u \| \leq M_2\}.$$

Similar to the proof of Lemma 3.5, we get that

Lemma 5.2. *Let the assumptions of lemma 5.1 hold, then for every $T > 0$, the mapping $(t, u) \mapsto S(t)u$ is Lipschitz continuous on $[0, T] \times \mathbb{B}$.*

Lemma 5.3. *Under the assumptions of lemma 5.1, $S(t_*) : \mathbb{B} \longrightarrow \mathbb{B}$ admits the following descomposition of the form*

$$S(t_*) = S_1 + S_2, \quad S_1 : \mathbb{B} \to \mathcal{H}_1, \quad S_2 : \mathbb{B} \to \mathcal{H}_2,$$

where S_1, S_2 satisfy

$$\| S_1(z_1) - S_1(z_2) \|_{\mathcal{H}_1} \leq \frac{1}{4} \| z_1 - z_2 \|_{\mathcal{H}_1}, \quad \forall z_1, z_2 \in \mathbb{B}, \tag{5.2}$$

and

$$\| S_2(z_1) - S_2(z_2) \|_{\mathcal{H}_2} \leq C \| z_1 - z_2 \|_{\mathcal{H}_1}, \quad \forall z_1, z_2 \in \mathbb{B}. \tag{5.3}$$

Proof. We take the following decomposition: denote by $S_1(t)u_0$ the solution of the linear homogeneous problem associated with(4.8) and set $S_2(t)u_0 = S(t)u_0 - S_1(t)u_0$. Then, for any two initial data $u_0^i \in \mathbb{B}$ and the corresponding solution $u^i(t) = S(t)u_0^i, \;\; i = 1, 2$, set $u_0 = u_0^1 - u_0^2$, we decompose $z^1(t) - z^2(t)$ into the sum

$$S(t)u_0^1 - S(t)u_0^2 = S_1(t)u_0^1 - S_1(t)u_0^2 + S_2(t)u_0^1 - S_2(t)u_0^2,$$

where $\tilde{v}(t) = S_1(t)u_0^1 - S_1(t)u_0^2$ solves the following linear homogeneous problem

$$\begin{cases} \tilde{v}_t - \Delta\tilde{v}_t - \Delta\tilde{v} = 0, \\ \tilde{v}(x,0) = u_0^1(x) - u_0^2(x), \end{cases} \tag{5.4}$$

where $\tilde{w}(t) = S_2(t)u_0^1 - S_2(t)u_0^2$ satisfy

$$\begin{cases} \tilde{w}_t - \Delta\tilde{w}_t - \Delta\tilde{w} + f(u_1) - f(u_2) = 0, \\ \tilde{w}(x,0) = 0. \end{cases} \tag{5.5}$$

Multply (5.4) by $\tilde{v}$ in $L^2(\Omega)$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\| \tilde{v} \|^2 + \| \nabla\tilde{v} \|^2) \quad + \quad \| \nabla\tilde{v} \|^2 = 0. \tag{5.6}$$

Using Poincar$\acute{e}$ inequalityand set $\alpha = \min\{1, \lambda_1\}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\| \tilde{v} \|^2 + \| \nabla\tilde{v} \|^2) + \alpha(\| \tilde{v} \|^2 + \| \nabla\tilde{v} \|^2) \leq C. \tag{5.7}$$

By the Growall lemma, and choosing $t_* \geq T_1$ large enough, we complete the proof of (5.2).
Multiply (5.5) by $-\Delta\tilde{w}$ in $L^2(\Omega)$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\| \nabla\tilde{w} \|^2 + \| \Delta\tilde{w} \|^2) + \| \Delta\tilde{w} \|^2 + \langle f(u^1) - f(u^2), -\Delta u\rangle = 0. \tag{5.8}$$

However

$$\begin{aligned} \langle f(u), -\Delta u\rangle &= \int_\Omega | f(u) || \Delta u | \,\mathrm{d}x \\ &\leq \frac{1}{2}\int_\Omega | f(u) |^2 \,\mathrm{d}x + \frac{1}{2} \| \Delta u \|^2 . \end{aligned} \tag{5.9}$$

By (F_2), we have
Case $1 (n > 4)$

$$\begin{aligned} \int_\Omega | f(u^1) - f(u^2) |^2 \,\mathrm{d}x &\leq \int_\Omega | f'(\theta u^1 - (1-\theta)u^2) |^2 | u^1 - u^2 |^2 \,\mathrm{d}x (0 \leq \theta \leq 1) \\ &\leq C\int_\Omega (1 + | \theta u^1 - (1-\theta)u^2 |^{\frac{8}{n-2}}) | u^1 - u^2 |^2 \,\mathrm{d}x \\ &\leq C \| u^1 - u^2 \|^2 + C \| \theta u^1 - (1-\theta)u^2 \|^2_{L^{2n/(n-4)}} \\ &\cdot \| u^1 - u^2 \|^2_{L^{2n(n-2)/(n^2-6n+16)}} . \end{aligned} \tag{5.10}$$

By the continuous embedding $\mathcal{H}_2 \hookrightarrow L^{2n/(n-4)}$ and the interpolation inequality $(0 < \frac{2n(n-2)}{n^2-6n+16} < \frac{2n}{n-4})$,

$$\| \cdot \|_{L^{2n(n-2)/(n^2-6n+16)}} \leq C_\varepsilon \| \cdot \|_{\mathcal{H}_1} + \varepsilon \| \cdot \|_{\mathcal{H}_2},$$

hence we have

$$\begin{aligned} \int_\Omega | f(u^1) - f(u^2) |^2 \,\mathrm{d}x &\leq C \| u^1 - u^2 \|^2 \\ &\quad + \frac{1}{2} \| \Delta(u^1 - u^2) \|^2 + C \| u^1 - u^2 \|^2_{\mathcal{H}_1} . \end{aligned} \tag{5.11}$$

Case 2 ($n = 3, 4$)
due to the Sobolev embedding

$$\mathcal{H}^2 \hookrightarrow L^{2p}(0 \leq p < \infty),$$

we can get a similar estimate as (5.11) easily.

According to the above estimates, for any $t > 0$, there exists increasing function $\mathcal{Q} : [0, \infty) \to [0, \infty)$ such that

$$\| \nabla \tilde{w} \|^2 + \| \Delta \tilde{w} \|^2 \leq \mathcal{Q}(t) \| z_1 - z_2 \|^2_{\mathcal{H}_1} . \tag{5.12}$$

that is

$$\| S_2(t)u_0^1 - S_2(t)u_0^2 \|_{\mathcal{H}_2} \leq \mathcal{Q}(t) \| z_1 - z_2 \|_{\mathcal{H}_1} . \tag{5.13}$$

The proof is complete.

Theorem 5.4. *(Exponential attractors in $H_0^1(\Omega)$ when $g(x) \in L^2(\Omega)$) Let Ω is a bounded open set of $\mathbb{R}^n (n \geq 3)$ with smooth boundary $\partial\Omega$, and f satisfy conditions $(F_1) - (F_2)$ and $g \in L^2(\Omega)$. Then the semigroup of operators $\{S(t)\}_{t\geq 0}$ generated by (1.1) − (1.3) has an exponential attractor $\mathcal{E}$ in $H_0^1(\Omega)$.*

Theorem 5.5. *(global attractor in $H_0^1(\Omega)$ when $g(x) \in L^2(\Omega)$) Let Ω is a bounded open set of $\mathbb{R}^n (n \geq 3)$ with smooth boundary $\partial\Omega$, and f satisfy conditions $(F_1) - (F_2)$ and $g \in L^2(\Omega)$. Then the semigroup of operators $\{S(t)\}_{t\geq 0}$ generated by (1.1) − (1.3) has a global attractor $\mathcal{A}$ in $H_0^1(\Omega)$, and $\mathcal{A}$ with finiteness of fractal dimension.*

REFERENCES

[1] Aifantis, E. C. (1980). On the problem of diffusion in solids. *Acta Mech.*, *37*, 265–296.

[2] Lions, J. L., & Magenes, E. (1972). *Non-homogeneous boundary value problems and applications*. Spring-verlag: Berlin.

[3] Kuttler, K., & Aifantis, E. C. (1987). Existence and uniqueness in nonclassical diffusion. *Quart. Appl. Math.*, *45*, 549–560.

[4] Kuttler, K., & Aifantis, E. C. (1988). Quasilinear evolution equations in nonclassical diffusion. *SIAM J. Math. Anal.*, *19*, 110–120.

[5] Aifantis, E. C. (2011). Gradient nanomechanics: Applications to deformation, fracture, and diffusion in nanopolycrystals. *Metall. Mater. Trans.*, *A42*, 2985–2998.

[6] Kalantarov, V. K. (1986). On the attractors for some non-linear problems of mathematical physics. *Zap. Nauch. Sem. LOMI*, *152*, 50–54.

[7] Xiao, Y. L. (2002). Attractors for a nonclassical diffusion equation. *Acta Math. Appl. Sinica.*, *18*, 273–276.

[8] Sun, C. Y., Wang, S. Y., & Zhong, C. K. (2007). Global attractors for a nonclassical diffusion equation. *Acta Math. Sinica, English Series.*, *23*, 1271–1280.

[9] Ma, Q. Z., & Zhong, C. K. (2004). Global attractors of strong solutions to nonclassical diffusion equations. *Journal of Lanzhou University*, *40*, 7–9.

[10] Wang, S. Y., Li, D. S., & Zhong, C. K. (2006). On the dynamics of a class of nonclassical parabolic equations. *J. Math. Anal. Appl.*, *317*, 565–582.

[11] Temam, R. (1997). *Infinite-dimensional dynamical systems in mechanics and physic*. New York: Springer.

[12] Sun, C. Y., & Yang, M. H. (2008). Dynamics of the nonclassical diffusion equations. *Asymptot. Anal.*, *59*, 51–81.

[13] Liu, Y. F., & Ma, Q. Z. (2009). Exponential attractors for a nonclassical diffusion equation. *Electronic Journal of Differential Equations*, *9*, 1–9.

[14] Ma, Q. Z., Liu, Y. F., & Zhang, F. H. (2012). Global attractors in $H^1(R^N)$ for nonclassical diffusion equations. *Discrete Dynamics in Nature and Society*, 1–17. doi:10.1155/2012/672762.

[15] Wu, H. Q., & Zhang, Z. Y. (2011). Asymptotic regularity for the nonclassical diffusion equation with lower regular forcing term. *Dynamical Systems: An International Journal*, *26*(4), 391–400.

[16] Zhao, T. G., Wu, Y. J., & Ma, H. P. (2012). Error analysis of Chebyshev-Legendre pseudo-spectral method for a class of nonclassical parabolic equation. *Journal of Scientific Computing*, *52*(3), 588–602.

[17] Zhao, T. G., Wu, Y. J., & Ma, H. P. (2010). Chebyshev-Legendre pseudospectral method for nonclassical parabolic equation. *Journal of Information and Computational Science*, *7*(8), 1809–1817.

[18] Sun, C. Y. (2010). Asymptotic regularity for some dissipative equations. *Journal of Differential Equations*, *248*(2), 342–362.

[19] Miranville, A., & Zelik, S. *Attractors for dissipative partial differential equations in bounded and unbounded domains* (Handbook of Differential Equations, Evolutionary Partial Differential Equations, C. M. Dafermos & M. Pokorny eds.). Amsterdam: Elsevier.

[20] Zelik, S. V. (2004). Asymptotic regularity of solutions of a nonautonomous damped wave equation with a critical growth exponent. *Comm. Pure Appl. Anal.*, *3*, 921–934.

[21] Zhang, F. H., & Liu, Y. F. (2014). Pullback attractors in $H^1(\mathbb{R}^N)$ for non-autonomous nonclassical diffusion equations. *Dynamical Systems*, *29*(1), 106–118.

[22] Pan, L. X., & Liu, Y. F. (2013). Robust exponential attractors for the non-autonomous nonclassical diffusion equation with memory. *Dynamical Systems*, *28*(4), 501–517.