CSCanada
Progress in Applied Mathematics
Vol. 7, No. 2, 2014, pp. 9-19
DOI: 10.3968/4098

ISSN 1925-251X [Print]
ISSN 1925-2528 [Online]
www.cscanada.net
www.cscanada.org

The Super-Convergence in Rheological Flow

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Received: October 14 2013/accepted: December 25 2013/Published online: April 24 2014

Abstract: To estimate the solution of the coupled first-order hyperbolic partial differential equations, we use both the boundary-layer method and numeric analysis to study the Cauchy fluid equations and *P-T/T* stress equation. On the macroscopic scale the free surface elements generate flow singularity and stress uncertainty by excessive tensile stretch. A numerical super-convergence semi-discrete finite element scheme is used to solve the time dependent equations. The coupled nonlinear solutions are estimated by boundary-layer approximation. Its numerical super convergence is proposed with the a priori and a posteriori error estimates.

Key words: Non-Newtonian fluid; Semi-discrete finite element method; Super convergence; Boundary-layer solution

Hou, L., Zhou, S. L., Sun, X. Y., Zhao, J. J., Qiu, L., & Li, H. L. (2014). The Super-Convergence in Rheological Flow. *Progress in Applied Mathematics, 7*(2), 145-19. Available from http://www.cscanada.net/index.php/pam/article/view/10.3968/4098 **DOI:** 10.3968/4098

1. INTRODUCTION

Inside the micro-scale materials Non-Newtonian fluid mechanics is another complex subject developed in recent decades to observe the fluid-structure interactions. It originated from polymer processing and involves many experimental fields. It is just an interdisciplinary subject, including mechanics, modern mathematics, chemical and engineering sciences, especially material science. Also, it is an important branch of modern fluid mechanics and is an important part of modern theology for virtual test. With further study for experts in the difficult and expensive chemical industry, oil, water conservancy, bioengineering, light industry and food material science, more realization of the computing accuracy impacted on the testing results become important issue in the complex testing conditions which is virtually designed by the CAD/CAE tools. The specific mathematical model is also widely used and promoted very rapidly to solve the practical aspect of the problems.

In this paper, fluid-solid coupling fractional differential equations: Cauchy equation (fluid equation) and *P-T/T* equation (solid equation) are used to describe the rheological process on non-Newtonian complex contact surface. The *P-T/T* equation adds exponential impact item to Maxwell equation which allows us to study the boundary-layer near discontinues interface to pin-point stress uncertainty and singularity. Rheological problems in cellular porous structural material are dealt with in detail in [1, 2] which describe the background of the deformable materials [5, 6].

2. MACROSCOPIC EQUATION

2.1 Fluid--Solid Coupling Equations

In the paper the contact surface (Hermit boundary conditions) and non-contact surface are taken as fluid composed of free surface element. In nature it is usually singular by excessive tensile stretch in the boundary-layer and randomly by the point force uncertainty. On the one hand, using standard variables of material science, the resistance of extensional and simple shear rate is analyzed by the characteristics of non-Newtonian *P-T/T* equation (1)[4, 8]. Coupled with Cauchy conservation equation (2)[3, 7], the elastic-plastic material deformation (shear thinning) caused by distribution changes in the macro-stress field τ can be calculated. Here we review Cauchy equation

$$\frac{\partial \underline{u}}{\partial t} = \frac{1}{\rho} \nabla \cdot \underline{\underline{\tau}} - \underline{u} \cdot \nabla \underline{u} \ , \tag{1}$$

and *P-T\T* equation which is used to calculate stress in the coupled equation [10],

$$\begin{aligned} \lambda \frac{\partial \underline{\underline{\tau}}}{\partial t} &= [2\eta \underline{\underline{D}} - \exp\left\{ \frac{\varepsilon \lambda}{\eta_0} \left(\tau_{xx} + \tau_{yy} \right) \right\} \underline{\underline{\tau}}] \\ &\quad - \lambda [\underline{u} \cdot \nabla \underline{\underline{\tau}} - \nabla \underline{u} \cdot \underline{\underline{\tau}} - \left(\nabla \underline{u} \cdot \underline{\underline{\tau}} \right)^T + \xi [\left(\underline{\underline{D}} \cdot \underline{\underline{\tau}} \right) + \left(\underline{\underline{D}} \cdot \underline{\underline{\tau}} \right)^T]] \end{aligned} \tag{2}$$

Where strain

$$D = \frac{1}{2} \ (\nabla u + \nabla u^T) \ .$$

2.2 Variation Principal Applied to the Coupled Equations

Boundary condition: Let Ω be smoothly bounded area in $R^{d,}$ with boundary Γ. Now we consider space-time domain $S=\Gamma\times(0,T)$, and apply no stress/strain on S, that is $u \cdot n|_{\Gamma_1} = 0, u|_{\Gamma_2} = 0$. The initial conditions are $u^0=u(x,0)$, $\tau^0=\tau(x, 0)$.

Let's introduce the following space,

$$X = H_0^1(\Omega)^2 = \left\{ x \in H^1(\Omega)^2,\ x \cdot n|_{\Gamma_1} = 0, x|_{\Gamma_2} = 0 \right\}$$
$$Y = H_0^1(\Omega)^{2\times 2} = \{ y \in H^1(\Omega)^{2\times 2}, y|_{\Gamma} = 0 \}$$

Define inner-product $(u, v) = \int_\Omega u : v \mathrm{d}x = \int_\Omega \sum u_i v_i \mathrm{d}x$, where *u*,*v* are vector or matrix. Therefore may define the following norm

$$\|v\|_0 = (\int_\Omega v : v\mathrm{d}x)^{\frac{1}{2}} .$$

Then we can prove that X and Y are Sobolev spaces.

We define the following tri-linear operator: B_1 (.,.,.), B_2 (.,.,.),b(.,.,.) are functional on subspaces $X\times X\times X$, $X\times Y\times Y$, $X\times Y\times Y$

$$B_1(u,v,w) = \int_\Omega u \cdot \nabla v : w\mathrm{d}x \quad \forall v,w \in X$$

$$B_2(u,\tau,v) = \int_\Omega u \cdot \nabla \tau : v\mathrm{d}x \quad \forall v \in Y$$

$$b(u,\tau,v) = \int_\Omega \left(1-\frac{\xi}{2}\right)[\nabla u \cdot \tau + (\nabla u \cdot \tau)^T] - \frac{\xi}{2}[\tau \cdot \nabla u + (\tau \cdot \nabla u)^T] : v\mathrm{d}x, \forall v \in Y$$

$$Q(w,\tau) = \int_\Gamma w_x \tau_{xx} \cos(n,x) + w_y \tau_{yy} \cos(n,y)\,\mathrm{d}s$$

Property:

B_1 (u,v,w),B_2 (u,τ,v) are continuous tri-linear form, and

$$|B_1(u,v,w)| \le C\|u\|\|v\|\|w\|$$
$$\le C(\|\nabla u\|\|v\|\|w\| + \|\nabla v\|\|u\|)\|w\|, \qquad \forall v,w \in Y$$
$$|B_2(u,\tau,v)| \le C\|u\|\tau\|w\|$$
$$\le C(\|\nabla u\|\|v\| + \|\nabla v\|\|u\|)\|\tau\|, \qquad \forall u \in X, \tau, v \in Y$$
$$|b(u,\tau,v)| \le C\|\nabla u\|\|\tau\|\|v\|, \qquad \forall u \in X, \tau, v \in Y$$

To solve coupled equations (1), (2), it is to find solution, satisfy

$$\begin{cases} (\underline{u}_t, w) = -\dfrac{1}{\rho}(\nabla w, \tau) + B_1(u,u,w) + Q(w,\tau) \\ \left(\underline{\underline{\tau}}_t, v\right) = \dfrac{2\eta}{\lambda}(D,v) - \dfrac{1}{\lambda}(\tau,v) - \dfrac{\varepsilon}{\eta_0}\left((\tau_{xx}+\tau_{yy})\tau, v\right) - B_2(u,\tau,v) + b(u,\tau,v) \end{cases} . \quad (3)$$

3. FINITE ELEMENT ANALYSIS TO THE COUPLED EQUATIONS

Making uniform rectangular elements in the space domain, with mesh size h we have the finite element space (X_h, Y_h), where $X_h=P_m(\Omega)^2 \cap X$,$Y_h=P_m(\Omega)^{(2\times2)} \cap Y$, surely .

Hypothesis 1, There is a mapping $X: u \to \bar{u} \in X_h$, satisfy $\left((\nabla(u-\bar{u}), v)\right) = 0$, and $\|u-\bar{u}\|_l \le Ch^{m+1-l}\|u\|_{m+1}, \quad l = 0,1$.

Hypothesis 2, There is an inverse inequality $\|\nabla u_h\| \le Ch^{-1}\|u_h\|$.

Hypothesis 3, There is finite element projection $\bar{u}$, satisfy $\left((\nabla(u-\bar{u}), v_h)\right) = 0$

The weak solution of the coupled equation is to find $(u_h, \tau_h) \in X_h \times Y_h$, for all satisfy

$$\begin{cases} (u_{h,t}\, w) = -\dfrac{1}{\rho}(\nabla w, \tau_h) + B_1(u_h, u_h, w) + Q(w, \tau_h) \\ (\tau_{h,t}, v) = \dfrac{2\eta}{\lambda}(D_h, v) - \dfrac{1}{\lambda}(\tau_h, v) - \left(\dfrac{\varepsilon}{\eta_0}(\tau_{xx} + \tau_{yy})\tau_h, v\right) + B_2(u_h, \tau_h, v) - b(u_h, \tau_h, v) \end{cases} \tag{4}$$

Theorem 1: The solution of the coupled equation , with m-th order FEA solution $(u_h,\ \tau_h)$ of the problem. There exist positive constant $c2$, for arbitrary t satisfy $\frac{1}{\lambda} + \frac{\varepsilon c_2}{\eta_0} + c_2 h^{-1} < T^{-\frac{3}{2}}$ we have the a priori estimate of convergence.

$$\| u - u_h \| \le \frac{c_1 T^{\frac{3}{2}} h^m + c_3 T^{\frac{3}{2}} h^{m-1}}{1 - \frac{1}{B}\left(\frac{C\eta h^{-2} T^3}{\rho\lambda} + \frac{C\eta h^{-3} T^3}{\lambda} + \frac{C h^{-1} T^3}{\rho} + C h^{-2} T^3\right)} + ch^m + ch^m$$

$$\| \tau - \tau_h \| \le ((1 + \frac{\eta}{\lambda}) \frac{C T^3 h^{m-1} + C T^3 h^{m-2}}{B - (\frac{C\eta h^{-2} T^3}{\rho\lambda} + \frac{C\eta h^{-3} T^3}{\lambda} + \frac{C h^{-1} T^3}{\rho} + C h^{-2} T^3)} + \frac{C T^{\frac{3}{2}} h^{m-1}}{B}$$

where $B = 1 - \left(\frac{1}{\lambda} - \frac{\varepsilon c_2}{\eta_0} - c_2 h^{-1}\right) T^{\frac{3}{2}}$, c, c_1, c_3 are positive constants.

proof: Note that $e_1 = u - u_h = u - \bar{u} + \bar{u} - u_h = e_1^* + \theta_1, e_2 = \tau - \tau_h$.

Operating (3) minus (4), we obtain

$$\begin{cases} (\theta_{1,t}, w) = -\dfrac{1}{\rho}(\nabla w, \tau - \tau_h) + B_1(u - u_h; u_h; w) + B_1(u; u - u_h; w) - (e_{1,t}^*, w) + Q(w, \tau - \tau_h) \\ (e_{2,t}, v) = \dfrac{2\eta}{\lambda}(D - D_h, v) - \dfrac{1}{\lambda}(\tau - \tau_h, v) - \dfrac{\varepsilon}{\eta_0}[((\tau_{xx} + \tau_{yy})(\tau - \tau_h) + (\tau_{xx} + \tau_{yy} - \tau_{xx,h} - \tau_{yy,h})\tau_h, v)] \\ \qquad + B_2(u_h, \tau - \tau_h, v) + b(u - u_h, \tau, v) + b(u_h, \tau - \tau_h, v) \end{cases}$$

Where $Q(w, \tau - \tau_h) = \int_\Gamma w_x(\tau_{xx} - \tau_{xx,h})\cos(n,x) + w_y(\tau_{yy} - \tau_{yy,h})\cos(n,y)\, ds$

$$\le \int_\Gamma w_y(\tau_{yy} - \tau_{xx} + \tau_{xx,h} - \tau_{yy,h})\cos(n,y)\, ds$$

from Holde's inequality,

$$\le \left[\int_\Gamma |w_y|^2 ds\right]^{1/2} \left[\int_\Gamma |\tau - \tau_h|^2 ds\right]^{1/2}$$

$$\le C\|w\|_1 \|\tau - \tau_h\|_1$$

and from Sobolev insertion theorem, with , $\left[\int_\Gamma |w_y|^2 ds\right]^{1/2}$,

we have

$$\begin{cases} (\theta_{1,t}, w) \le -\dfrac{1}{\rho}(\nabla w, \tau - \tau_h) + B_1(u - u_h; u_h; w) + B_1(u; u - u_h; w) - (e_{1,t}^*, w) + C\|w\|_1\|\tau - \tau_h\|_1 \\ (e_{2,t}, v) = \dfrac{2\eta}{\lambda}(D - D_h, v) - \dfrac{1}{\lambda}(\tau - \tau_h, v) - \dfrac{\varepsilon}{\eta_0}[(\tau_{xx} + \tau_{yy})(\tau - \tau_h) + (\tau_{xx} + \tau_{yy} - \tau_{xx,h} - \tau_{yy,h})\tau_h, v)] \\ \qquad +B_2(u_h, \tau - \tau_h, v) + b(u - u_h, \tau, v) + b(u_h, \tau - \tau_h, v) \end{cases}$$

Let $w = \theta_{1,t}$, $v = e_{2,t}$

$$\begin{cases} (\theta_{1,t}, \theta_{1,t}) = -\dfrac{1}{\rho}(\nabla\theta_{1,t}, e_2) + B(e_1, u_h, \theta_{1,t}) + B(u; \theta_1, \theta_{1,t}) - (e_{1,t}^*, \theta_{1,t}) + C\|\theta_{1,t}\|_1\|e_2\|_1 \\ (e_{2,t}, \theta_{2,t}) = \dfrac{2\eta}{\lambda}(D(\theta_1), e_{2,t}) - \dfrac{1}{\lambda}(e_2, e_{2,t}) - \dfrac{\varepsilon}{\eta_0}[(\tau_{xx} + \tau_{yy})e_2 + (\tau_{xx} + \tau_{yy} - \tau_{xx,h} - \tau_{yy,h})\tau_h, e_{2,t})] \\ \qquad +B(e_1, \tau, e_{2,t}) + B(u_h, e_2, e_{2,t}) + b(e_1, \tau, e_{2,t}) + b(u_h, e_2, e_{2,t}) \end{cases}$$

Use of the property in hypothesis 1, 2, 3 yields

$$\|\theta_{1,t}\|^2 \le \frac{ch^{-1}}{\rho}\|\theta_{1,t}\|\cdot\|e_2\| + ch^{-1}(\|u_h\| + \|u\|)\|\theta_{1,t}\|\cdot\|e_1\| + \|e_{1,t}^*\|\cdot\|\theta_{1,t}\| + Ch^{-2}\|e_2\|$$

$$\|e_{2,t}\|^2 \le \frac{ch^{-1}\eta}{\lambda}\|\theta_1\|\cdot\|e_{2,t}\| + \frac{1}{\lambda}\|e_2\|\cdot\|e_{2,t}\| + \frac{c\varepsilon}{\eta_0}(\|\tau\| + \|\tau_h\|)\|e_2\|\cdot\|e_{2,t}\| + ch^{-1}[\|\tau\|\cdot\|e_1\|\cdot\|e_{2,t}\| + \|u_h\|\cdot\|e_2\|\cdot\|e_{2,t}\| + \|\theta_1\|\cdot\|\tau\|\cdot\|e_{2,t}\|]$$

Eliminating the common factors on both sides, yield

$$\|\theta_{1,t}\| \le \frac{ch^{-1}}{\rho}\|e_2\| + ch^{-1}(\|u_h\| + \|u\|)\|e_1\| + \|e_{1,t}^*\| \le \frac{c}{\rho}\|e_2\| + c_1h^{-1}\|e_1\| + \|e_{1,t}^*\| + Ch^{-2}\|e_2\|$$

$$\|e_{2,t}\| \le \frac{ch^{-1}\eta}{\lambda}\|\theta_1\| + \frac{1}{\lambda}\|e_2\| + \frac{c\varepsilon}{\eta_0}(\|\tau\| + \|\tau_h\|)\|e_2\| + ch^{-1}[\|\tau\|\cdot\|e_1\| + \|u_h\|\cdot\|e_2\| + \|\theta_1\|\cdot\|\tau\|]$$

$$\|e_{2,t}\| \le \frac{ch^{-1}\eta}{\lambda}\|\theta_1\| + \frac{1}{\lambda}\|e_2\| + \frac{\varepsilon c_2}{\eta_0}\|e_2\| + c_2h^{-1}[\|e_1\| + \|\theta_1\|] + c_2h^{-1}\|e_2\|$$

use of $\|\theta(t)\|^2 \le T\int_0^t \|\theta_t\|^2\, dt$, obtain

$$\|\theta_1(t)\| \le \left(\frac{Ch^{-1}T^{\frac{3}{2}}}{\rho} + Ch^{-2}T^{\frac{3}{2}}\right)\|e_2\| + ch^{-1}T^{\frac{3}{2}}\|e_1\| + cT^{\frac{3}{2}}h^m, \tag{5}$$

$$\|e_2(t)\| \le \frac{ch^{-1}\eta T^{\frac{3}{2}}}{\lambda}\|\theta_1\| + \left(\frac{1}{\lambda} + \frac{\varepsilon c_2}{\eta_0} + c_2h^{-1}\right)T^{\frac{3}{2}}\|e_2\| + c_2h^{-1}T^{3/2}[\|e_1\| + \|\theta_1\|] \tag{6}$$

$$\left[1 - \left(\frac{1}{\lambda} - \frac{\varepsilon c_2}{\eta_0} - c_2h^{-1}\right)T^{\frac{3}{2}}\right]\|e_2(t)\| \le \frac{c\eta h^{-1}T^{3/2}}{\lambda}\|\theta_1\| + c_2h^{-1}T^{3/2}[\|e_1\| + \|\theta_1\|]\ . \tag{7}$$

Let $1 - \left(\frac{1}{\lambda} - \frac{\varepsilon c_2}{\eta_0} - c_2h^{-1}\right)T^{\frac{3}{2}} = B$

Operating (7) substitute into (5), and use of $\|e_1\| \le \|\theta_1\| + ch^m$

$$\| \theta_1(t) \| \le \frac{1}{B}\left(\frac{Ch^{-1}T^{\frac{3}{2}}}{\rho} + Ch^{-2}T^{\frac{3}{2}}\right)\left\{\frac{c\eta h^{-1}T^{\frac{3}{2}}}{\lambda} \| \theta_1 \| + c_2 h^{-1}T^{\frac{3}{2}}[\| e_1 \| + \| \theta_1 \|]\right\}$$

$$+ch^{-1}T^{3/2} \| e_1 \| + cT^{\frac{3}{2}}h^m$$

$$\le \frac{1}{B}\left(\frac{Ch^{-1}T^{\frac{3}{2}}}{\rho} + Ch^{-2}T^{\frac{3}{2}}\right)\left[\frac{C\eta h^{-1}T^{\frac{3}{2}}}{\lambda} \| \theta_1 \| + CT^{\frac{3}{2}} \| \theta_1 \|\right] + CT^{\frac{3}{2}}h^m + CT^{\frac{3}{2}}h^{m-1}$$

therefore

$$\| \theta_1(t) \| \le \frac{CT^{\frac{3}{2}}h^m + CT^{\frac{3}{2}}h^{m-1}}{1 - \frac{1}{B}\left(\frac{C\eta h^{-2}T^3}{\rho\lambda} + \frac{C\eta h^{-3}T^3}{\lambda} + \frac{Ch^{-1}T^3}{\rho} + Ch^{-2}T^3\right)}$$

$$\| e_1(t) \| \le \| \theta_1(t) \| + ch^m \le \frac{c_1 T^{\frac{3}{2}}h^m + c_3 T^{\frac{3}{2}}h^{m-1}}{1 - \frac{1}{B}\left(\frac{C\eta h^{-2}T^3}{\rho\lambda} + \frac{C\eta h^{-3}T^3}{\lambda} + \frac{Ch^{-1}T^3}{\rho} + Ch^{-2}T^3\right)} + ch^m \tag{8}$$

Operating (8) substitute into (7) obtain $\| e_2(t) \|$.

$$\| e_2(t) \| \le \left(1 + \frac{\eta}{\lambda}\right)\frac{CT^3h^{m-1} + CT^3h^{m-2}}{B - \left(\frac{C\eta h^{-2}T^3}{\rho\lambda} + \frac{C\eta h^{-3}T^3}{\lambda} + \frac{Ch^{-1}T^3}{\rho} + Ch^{-2}T^3\right)} + \frac{CT^{\frac{3}{2}}h^{m-1}}{B}.$$

4. EULER FORMULATION UPON THE TIME DOMAIN

For the stress, strain change upon TIME, use of finite difference method, with sub-intervals $0=t_0<t_1<\cdots\cdots<t_N=T$ in time domain (0,T], note $I_n=(t_{n-1},t_n)$,$k_n=t_n-t_{n-1}$ is time step length, $k=\max_n k_n$, note U^n,τ^n are the approximate time dependent boundary value of the exact solution $u(x,t)$,$\tau(x,t)$ on the mesh grid t_n,$n=1,2\cdots N$. Note that the initial value is bounded:

$$\left\| u_h^0 - u^0 \right\|_l \le Ch^{m+1-l}\| u^0 \|_{m+1}, \quad l = 0,1$$

$$\left\| \tau_h^0 - \tau^0 \right\|_l \le Ch^{m+1-l}\| \tau^0 \|_{m+1}, \quad l = 0,1$$

In time domain, by use of Euler backward finite difference, to solve $(U^n, \tau^n) \in (X_h, Y_h)$, $\quad n = 1,2\cdots N$.

$$\begin{cases} \left(\dfrac{U^n - U^{n-1}}{k_n}, w\right) = -\dfrac{1}{\rho}(\nabla w, \tau^n) + B_1(U^n, U^n, w) + Q(w, \tau^n) \\ \left(\dfrac{\tau^n - \tau^{n-1}}{k_n}, v\right) = \dfrac{2\eta}{\lambda}(D(U^n), v) - \dfrac{1}{\lambda}(\tau^n, v) \\ -\dfrac{\varepsilon}{\eta_0}\left(\left({\tau_{xx}}^n + {\tau_{yy}}^n\right)\tau^n, v\right) - B_2(U^n, \tau^n, v) + b(U^n, \tau^n, v) \end{cases} \tag{9}$$

Further simplification $u = u(t_n)$, $\tau = \tau(t_n)$, yields the ,from (3) and (9), as

$$\begin{cases} \left(\frac{U^n - U^{n-1}}{k_n} - u_t, w\right) = -\frac{1}{\rho}(\nabla w, \tau^n - \tau) + B_1(U^n - u, u, w) + B_1(U^n, U^n - u, w) + Q(w, \tau^n - \tau) \\ \left(\frac{\tau^n - \tau^{n-1}}{k_n} - \tau_t, v\right) = \frac{2\eta}{\lambda}(D(U^n - u), v) - \frac{1}{\lambda}(\tau^n - \tau, v) \\ -\frac{\varepsilon}{\eta_0}\left(\left({\tau_{xx}}^n + {\tau_{yy}}^n\right)\tau^n - \left(\tau_{xx} + \tau_{yy}\right)\tau, v\right) + B_2(u - U^n, \tau, v) + B_2(U^{nh}, \tau - \tau^n, v) \\ +b(U^n - u, \tau, v) + b(U^n, \tau^n - \tau, v) \end{cases}$$

Theorem 2: Let u, τ be the exact solution of the equation, and $\{U^n, \tau^n\}$ be approximate solution defined by (4), then a posteriori error estimate

$$\left(q^n - \frac{C(h^{-2}k^2 + \rho k h^{-3})(\lambda + \eta)}{(1 - C_1 h^{-1} k)\rho\lambda}\right)\|\tau^n - \tau(t_n)\|$$
$$\leq Ck\int_0^{t_n}\|\tau_t(s)\|\mathrm{d}s + \frac{Ch^{-1}k(\lambda + \eta)}{(1 - C_1 h^{-1} k)\lambda}\left(k\int_0^{t_n}\|u_t(s)\|\mathrm{d}s + Ch^{m+1}\|u^0\|_{m+1}\right)$$
$$+ Ch^{m+1}\|\tau^0\|_{m+1}$$
$$\left((1 - C_1 h^{-1} k)^n - \frac{Ch^{-2}k^2(\lambda + \eta)}{\rho q \lambda}\right)\|U^n - u(t_n)\|$$
$$\leq \mathrm{Ck}\int_0^{t_n}\|\mathrm{u}_t(s)\|\mathrm{d}s + \frac{C(h^{-1}k + \rho h^{-2})}{\rho q}\left(k\int_0^{t_n}\|\tau_t(s)\|\mathrm{d}s + Ch^{m+1}\|\tau^0\|_{m+1}\right)$$
$$+ Ch^{m+1}\|u^0\|_{m+1}$$

Where C_1 satisfy $C_1 h^{-1} k < 1$

proof:

Let

$$e^n = U^n - u(t_n),\quad \bar{\partial}_t e^n = \frac{e^n - e^{n-1}}{k_n},\ \theta^n = \tau^n - \tau(t_n),\quad \bar{\partial}_t \theta^n = \frac{\theta^n - \theta^{n-1}}{k_n}$$

$k = \max_n k_n$.

We first study the Cauchy equation

$$\left(\bar{\partial}_t e^n, w\right) = -\frac{1}{\rho}(\nabla w, \tau^n - \tau) + B_1(U^n - u, u, w) + B_1(U^n, U^n - u, w) - (R^n, w)\ .$$

Where $R^n = \frac{u(t_n) - u(t_{n-1})}{k_n} - u_t(t_n) = \bar{\partial}_t u(t_n) - u_t(t_n)$.

Let w=e^n nd denoted that $R^j = \frac{u(t_j) - u(t_{j-1})}{k_j} - u_t(t_j) = -k_j^{-1}\int_{t_{j-1}}^{t_j}(s - t_{j-1})u_{tt}(s)\mathrm{d}s$

$$\sum_{j=1}^{n} k_j\|R^j\| \leq \sum_{j=1}^{n}\left\|\int_{t_{j-1}}^{t_j}(s - t_{j-1})u_{tt}(s)\mathrm{d}s\right\| \leq \mathrm{Ck}\int_0^{t_n}\|u_{tt}(s)\|\mathrm{d}s\ \left(\bar{\partial}_t e^n, e^n\right)$$
$$= \frac{1}{k_n}(\|e^n\|^2 - (e^{n-1}, e^n))$$

following the same procedure as above, we obtain

$$\|e^n\|^2 \le \|e^{n-1}\|\|e^n\| + k_n\|R^n\|\|e^n\| + \frac{C_1h^{-1}k_n}{\rho}\|e^n\|\|\theta^n\| + C_1h^{-1}k_n\|e^n\|^2 + Ch^{-2}\|\theta^n\|\|e^n\|$$

where the constant satisfy $C_1h^{-1}k<1$ thus

$$\|e^n\| \le \|e^{n-1}\| + k_n\|R^n\| + \frac{C_1h^{-1}k_n}{\rho}\|\theta^n\| + C_1h^{-1}k_n\|e^n\| + Ch^{-2}\|\theta^n\|$$

$$(1-C_1h^{-1}k)\|e^n\| - \|e^{n-1}\| \le k_n\|R^n\| + \frac{C_1h^{-1}k_n + C\rho h^{-2}}{\rho}\|\theta^n\|$$

Continue the recursive form yields

$$(1-C_1h^{-1}k)^n\|e^n\| - \|e^0\| \le \sum_{j=1}^{n}(1-C_1h^{-1}k)^{j-1}k_j\|R^j\| + \sum_{j=1}^{n}(1-C_1h^{-1}k)^{j-1}\frac{C_1h^{-1}k_n + C\rho h^{-2}}{\rho}\|\theta^j\|$$

$$\le \sum_{j=1}^{n}k_j\|R^j\| + \frac{C_1h^{-1}k_n + C\rho h^{-2}}{\rho}\|\theta^n\|$$

as $R^j = \frac{u(t_j)-u(t_{j-1})}{k_j} - u_t(t_j) = -k_j^{-1}\int_{t_{j-1}}^{t_j}(s-t_{j-1})u_{tt}(s)\mathrm{d}s$

$$\sum_{j=1}^{n}k_j\|R^j\| \le \sum_{j=1}^{n}\left\|\int_{t_{j-1}}^{t_j}(s-t_{j-1})u_{tt}(s)\mathrm{d}s\right\| \le Ck\int_0^{t_n}\|u_{tt}(s)\|\mathrm{d}s$$

therefore the *a* posteriori error estimate for *u* is

$$(1-C_1h^{-1}k)^n\|e^n\| - \|e^0\| \le Ck\int_0^{t_n}\|u_{tt}(s)\|\mathrm{d}s + \frac{C_1h^{-1}k_n + C\rho h^{-2}}{\rho}\|\theta^n\|. \quad (10)$$

Secondly we study the *PTT* equation

$$\left(\bar{\partial}_t\theta^n, v\right) = \frac{2\eta}{\lambda}(D(e^n), v) - \frac{1}{\lambda}(\theta^n, v) - \frac{\varepsilon}{\eta_0}\left(\left(\tau_{xx}{}^n + \tau_{yy}{}^n\right)\tau^n - \left(\tau_{xx} + \tau_{yy}\right)\tau, v\right)$$

$$B_2(u-U^n,\tau,v) + B_2(U^n,\tau-\tau^n,v) + b(U^n-u,\tau,v) + b(U^n,\tau^n-\tau,v) - (R^n,w)$$

where $R^n = \dfrac{\tau(t_n)-\tau(t_{n-1})}{k_n} - \tau_t(t_n) = \bar{\partial}_t\tau(t_n) - \tau_t(t_n)$.

Let $v=\theta^n$,

as $\left(\bar{\partial}_t\theta^n, \theta^n\right) = \frac{1}{k_n}(\|\theta^n\|^2 - (\theta^{n-1},\theta^n))$, repeat same analysis as above, arrives

$$\frac{1}{k_n}\|\theta^n\|^2 \le \frac{C_2\eta h^{-1}}{\lambda}\|e^n\|\|\theta^n\| + (\frac{C_2\varepsilon}{\eta_0} - \frac{1}{\lambda})\|\theta^n\|^2 + C_2h^{-1}(\|e^n\| + \|\theta^n\|)\|\theta^n\| + \frac{1}{k_n}\|\theta^n\|\|\theta^{n-1}\| + \|\theta^n\|\|R^n\|$$

$$\|\theta^n\| \le k_n\left(\frac{C_2\eta h^{-1}}{\lambda}\|e^n\| + \left(\frac{C_2\varepsilon}{\eta_0} - \frac{1}{\lambda}\right)\|\theta^n\| + C_2h^{-1}(\|e^n\| + \|\theta^n\|) + \|R^n\|\right) + \|\theta^{n-1}\|$$

Simplification leads to

$$\left(1-k_n\left(\frac{C_2\varepsilon}{\eta_0}-\frac{1}{\lambda}+C_2h^{-1}\right)\right)\|\theta^{\mathrm{n}}\|-\|\theta^{n-1}\|\leq k_n\|R^n\|+(C_2h^{-1}+\frac{C_2\eta h^{-1}}{\lambda})k_n\|e^n\|$$

where $k\left(\frac{C_2\varepsilon}{\eta_0}-\frac{1}{\lambda}+C_2h^{-1}\right)<1$, let $q=1-k_n\left(\frac{C_2\varepsilon}{\eta_0}-\frac{1}{\lambda}+C_2h^{-1}\right)$

Continue the recursive form yields

$$q^n\|\theta^n\|-\|\theta^0\|\leq\sum_{j=1}^{n}q^{j-1}\,k_j\left\|R^j\right\|+\sum_{j=1}^{n}q^{j-1}\,(C_2h^{-1}+\frac{C_2\eta h^{-1}}{\lambda})k_n\left\|e^j\right\|$$

$$\leq\sum_{j=1}^{n}k_j\left\|R^j\right\|+Ch^{-1}(1+\frac{\eta}{\lambda})k\|e^n\|$$

As $R^j=\frac{\tau(t_j)-\tau(t_{j-1})}{k_j}-\tau_t(t_j)=-k_j^{-1}\int_{t_{j-1}}^{t_j}(s-t_{j-1})\tau_{tt}(s)\mathrm{d}s$

$$\sum_{j=1}^{n}k_j\left\|R^j\right\|\leq\sum_{j=1}^{n}\left\|\int_{t_{j-1}}^{t_j}(s-t_{j-1})\tau_{tt}(s)\mathrm{d}s\right\|\leq \mathrm{Ck}\int_0^{t_n}\|\tau_{tt}(s)\|\mathrm{d}s$$

therefore

$$q^n\|\theta^n\|-\|\theta^0\|\leq \mathrm{Ck}\int_0^{t_n}\|\tau_{tt}(s)\|\mathrm{d}s+Ch^{-1}\left(1+\frac{\eta}{\lambda}\right)k\|e^n\| \quad . \tag{11}$$

Finally, equations(10) and(11)yield the a posteriori error estimate for

$$\left(q^n-\frac{C(h^{-2}k^2+\rho kh^{-3})(\lambda+\eta)}{(1-C_1h^{-1}k)\rho\lambda}\right)\|\theta^n\|-\|\theta^0\|$$

$$\leq Ck\int_0^{t_n}\|\tau_{tt}(s)\|\mathrm{d}s+\frac{Ch^{-1}k(\lambda+\eta)}{(1-C_1h^{-1}k)\lambda}(\,k\int_0^{t_n}\|u_{tt}(s)\|\mathrm{d}s+\|e^0\|)$$

$$\left((1-C_1h^{-1}k)^n-\frac{Ch^{-2}k^2(\lambda+\eta)}{\rho q\lambda}\right)\|e^n\|-\|e^0\|$$

$$\leq Ck\int_0^{t_n}\|u_{tt}(s)\|\mathrm{d}s+\frac{C(h^{-1}k+\rho h^{-2})}{\rho q}(k\int_0^{t_n}\|\tau_{tt}(s)\|\mathrm{d}s+\|\theta^0\|)$$

The proof is complete.

5. ESTIMATES OF THE CE SOLUTION

For the evaluation of the coupled equation CE we replacing with and replacing small with a constant . We have normalized internal boundary-layer equation[10]

$$y''+x^2y'+2xy+a(P+Nx^2)\,x^2=0,\quad y(\pm\infty)=0.$$

For arbitrary constant a, the evaluation of the CE property, without the asymptotic estimates the analysis has been carried out as follows

Integrating $\int \frac{\mathrm{d}}{\mathrm{d}x}\left(a\frac{\mathrm{d}y}{\mathrm{d}x}+x^2y\right)\mathrm{d}x=\int -ax^2(P+Nx^2)\mathrm{d}x$

Yields $a\frac{\mathrm{d}y}{\mathrm{d}x}+x^2y=-\frac{1}{5}aNx^5-\frac{1}{3}aPx^3+c_1$

Obtain the first-order equation $\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{x^2y}{a}=-\frac{\frac{1}{5}aNx^5+\frac{1}{3}aPx^3-c_1}{a}$

Multiply integrand on both sides $\mu(x)=e^{\int\frac{x^2}{a}\mathrm{d}x}=e^{\frac{x^3}{3a}}$

To the boundary layer equation $e^{\frac{x^3}{3a}}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{e^{\frac{x^3}{3a}}x^2}{a}y=-\frac{e^{\frac{x^3}{3a}}\left(\frac{1}{5}aNx^5+\frac{1}{3}aPx^3-c_1\right)}{a}$

Let $-\frac{t^3}{3a}=u$.

Then finally yields algebraic modified exponential solution as the characteristic of CE

$$y=\left(c_2+3^{-\frac{5}{3}}a^{\frac{1}{3}}(aP+3c_1)\Gamma\left(\frac{1}{3},-\frac{x^3}{3a}\right)\right)e^{-\frac{x^3}{3a}}+\frac{3}{5}aN-\frac{1}{5}Nx^3-\frac{1}{3}Px\,.$$

The exponential contribution towards the boundary-layer eigen-solution for small $a=\varepsilon$,

$$h(k)=\left(c_2+3^{-\frac{5}{3}}\varepsilon^{\frac{1}{3}}(\varepsilon P+3c_1)\Gamma\left(\frac{1}{3},-\frac{k^3}{3\varepsilon}\right)\right)e^{-\frac{k^3}{3\varepsilon}}+\frac{3}{5}\varepsilon^2N-\frac{1}{3}\varepsilon Pk-\frac{1}{5}\varepsilon Nk^3$$

$$=c_2+3^{-\frac{2}{3}}\Gamma\left(\frac{1}{3}\right)c_1\varepsilon^{\frac{1}{3}}+3^{-\frac{5}{3}}\Gamma\left(\frac{1}{3}\right)P\varepsilon^{\frac{4}{3}}+\frac{3}{5}N\varepsilon^2, k\to 0$$

Is in comparison to the internal solution $h(0-)=3^{-\frac{2}{3}}\Gamma\left(\frac{1}{3}\right)c_1\varepsilon^{\frac{1}{3}}+O\left(P\varepsilon^{\frac{5}{3}},N\varepsilon^{\frac{7}{3}}\right)$ as previously estimated[2], and this concludes the general behavior of the CE solution in the internal boundary layer.

6. NUMERICAL EXPERIMENT

Equipped with nine point bi-quadratic polynomial as FEA base we solve the coupled equation in the FEA iterative computing. Then a numerical scheme in time domain has been carried out based on Euler FDM. In the pre-processing mesh, equidistant grid points both in space and time are constructed. Because of complex nature of the coupled equation, it is very difficult to obtain the exact solution and convergence of . Instead, we solve for the numerical results, the a posteriori error estimate indicated in the following Table 1.

Table 1
Error Analysis

Ite / Nodes	6th	7th	8th	10th	11th
1	0.00033708	0.00022998	0.0002676	0.00057429	0.00037569
2	8.8208×10^{-5}	4.5595×10^{-5}	4.6181×10^{-5}	8.0985×10^{-5}	4.876×10^{-5}

To be continued

Continued

3	-5.0693×10^{-5}	-2.056×10^{-5}	-1.8412×10^{-5}	-3.0782×10^{-5}	-1.3676×10^{-5}
4	8.7456×10^{-5}	6.674×10^{-5}	0.00011358	0.00023398	0.0003104
5	5.9978×10^{-5}	6.9437×10^{-5}	7.4428×10^{-5}	9.1437×10^{-5}	9.1691×10^{-5}
6	-3.5486×10^{-5}	-3.3895×10^{-5}	-3.0114×10^{-5}	-2.8555×10^{-5}	-2.046×10^{-5}
7	-1.8524×10^{-5}	4.1078×10^{-5}	3.4175×10^{-5}	3.9823×10^{-5}	0.00013488
8	3.5158×10^{-5}	3.7394×10^{-5}	4.0822×10^{-5}	3.1552×10^{-5}	5.5295×10^{-5}
9	-2.148×10^{-5}	$- 2.3729\times10^{-5}$	-2.4226×10^{-5}	-1.5397×10^{-5}	-2.88×10^{-5}

From the result in the table, errors bounded under 0.0001.

According to the above theoretical analysis, with 3rd order convergence, it shows the consistent comparison of the theorem and numeric test.

7. SUMMARY

This article uses the macro CE equation to study the solutions of non-Newtonian flow. Using fluid-structure coupled nonlinear partial differential equations we may describe the non-Newtonian fluid deformation and its stress changes. These data are not smoothed numerically spread out the region, as the natural solution has not been polished. Under certain smoothness conditions, we have its convergence and general solution estimates for CE scheme.

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