CSCanada

Progress in Applied Mathematics
Vol. 7, No. 1, 2014, pp. [9–19]
DOI: 10.3968/3002

ISSN 1925-251X [Print]
ISSN 1925-2528 [Online]
www.cscanada.net
www.cscanada.org

Pseudo-Parallel Legendrian Submanifolds With Flat Normal Bundle of Sasakian Space Forms

CHEN Xiaomin[a],* and YANG Xue[b],[c]

[a] College of Sciences, China University of Petroleum (Beijing), Beijing, China.
[b] Institute of Remote Sensing Applications Chinese Academy Of Sciences, Beijing, China.
[c] School of Computer and Information Engineering, Henan University, Kaifeng, China.

* Corresponding author.
Address: College of Science, China University of Petroleum (Beijing), Beijing; E-Mail: xmchen1983@126.com

Supported by The NNSF(11071257), partially by Science Foundation of China University of Petroleum (Beijing) and the NNSF of China (41201349).

Received: October 14, 2013 / Accepted: December 20, 2013 / Published online: January 24, 2014

Abstract: Let M^n be a Legendrian submanifold with flat normal bundle of a Sasakian space form $\widetilde{M}^{2n+1}(c)$. Further, M^n is said to be pseudo-parallel if its second fundamental form h satisfies $\overline{R}(X,Y)\cdot h = L(X\wedge Y\cdot h)$. In this article we shall prove that M is semi-parallel or totally geodesic and if M satisfies $L\neq \frac{c+3}{4}$ then it is minimal in case of $n\geq 2$. Moreover, we show that if M^n is also a H-umbilical submanifold then either M^n is $L=\frac{c+3}{4}$, or $n=1$.

Key words: Legendrian submanifold; Minimal submanifold; H-umbilical submanifold; Pseudo-parallel submanifold; Sasakian space form

Chen X., & Yang X. (2014). Pseudo-Parallel Legendrian Submanifolds With Flat Normal Bundle of Sasakian Space Forms. *Progress in Applied Mathematics, 7*(1), 9–19. Available from: URL: http://www.cscanada.net/index.php/pam/article/. DOI: http://dx.doi.org/10.3968/3002

1. INTRODUCTION

Recall that an isometric immersion $f : M^n \to \widetilde{M}^{n+k}$ from an n-dimensional Riemannian manifold into $n+k$-dimensional Riemannian with metric g is *pseudo-parallel* if its second fundamental form h satisfies

$$\overline{R}(X,Y)\cdot h = LX\wedge Y\cdot h, \tag{1.1}$$

where $\overline{R}(X,Y)$ is the curvature operator with respect to the van der Waerden-Bortolotti connection $\overline{\nabla}$ of f, L is some suitable smooth function on M and $X\wedge Y$ is an operator: $(X\wedge Y)Z = g(X,Z)Y - g(Y,Z)X$. So M is also referred as a L-pseudo-parallel submanifold of $\widetilde{M}$. In particular, if $L\equiv 0$, M is called a *semi-parallel submanifold*.

In fact, the definition of pseudo-parallel was introduced in [1],[2] as an natural extension of semi-parallel and as the extrinsic analogue of pseudo-symmetry in the sense of Deszcz [7], i.e., the curvature operator of a semi-Riemannian manifold(M,g) satisfies

$$R(X,Y)\cdot R = L_R X\wedge Y\cdot R,$$

for any X,Y tangent to M, L_R being some real value function on M.

Recently, concerning the study of pseudo-parallel immersion there are many results (see [1,2,9–11]), where the ambient manifold $\widetilde{M}$ has constant sectional curvature. In particular, we observe that Chacón and Lobos [5] studied pseudo-parallel Lagrangian submanifolds in a complex space form, and gave several properties. Also, they proved a local classification of pseudo-parallel Lagrangian surfaces. Analogous to the Lagrangian submanifolds in a complex space form, we consider M^n is a Legendrian submanifold in Sasakian space form. Such a submanifold has been deeply studied over the past of several decades. However, for the pseudo-parallel submanifolds in a Sasakian space form, there is only A.Yildiz etc's result [14], where they considered pseudo-parallel C-totally real minimal submanifolds in a Sasakian space form $\widetilde{M}^{2n+1}(c)$ and showed it is totally geodesic if $Ln - \frac{1}{4}(n(c+3)+c-1)\geq 0$. In the present paper we consider the Legendrian submanifolds with flat normal bundle in Sasakian space forms, which satisfy pseudo-parallel condition (1.1).

In section 2 we introduce some necessary basic conceptions and give some properties. The section 3 is our main results.

2. BASIC CONCEPTS

2.1. Sasakian Space Form

Let $\widetilde{M}^{2n+1}$ be a $2n+1$-dimensional Riemannian manifold. $\widetilde{M}$ is called an *almost contact manifold* if it is equipped with an almost contact structure (ϕ,ξ,η), where ϕ is a $(1,1)$-tensor field, ξ a unit vector field, η a one-form dual to ξ satisfying

$$\phi^2 = -I+\eta\otimes\xi,\ \eta\circ\phi = 0,\ \phi\circ\xi = 0. \tag{2.2}$$

It is well-known that there exists a Riemannian metric $\widetilde{g}$ such that

$$\widetilde{g}(\phi X, \phi Y) = \widetilde{g}(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$\widetilde{g}(\phi X, Y) = -\widetilde{g}(X, \phi Y),\ \widetilde{g}(X, \xi) = \eta(X), \tag{2.4}$$

where $X, Y \in \mathfrak{X}(\widetilde{M})$. Moreover, if the almost contact structure (ϕ, ξ, η) is normal, i.e.

$$(\widetilde{\nabla}_X \phi)Y = \widetilde{g}(X, Y)\xi - \eta(Y)X, \quad \widetilde{\nabla}_X \xi = -\phi X,$$

for any vectors X, Y on $\widetilde{M}$, where $\widetilde{\nabla}$ denotes the connection with respect to $\widetilde{g}$, then $\widetilde{M}$ is said to be a *Sasakian manifold*. For more details and background, see [4] and [13].

A plane of $T_p\widetilde{M}$ at p is called *ϕ-section* if it is spanned by X and ϕX, where X is orthonormal to ξ. The curvature of ϕ-section is called *ϕ-sectional curvature*.

A $2n+1$-Sasakian space form is defined as a $2n+1$-Sasakian manifold with constant ϕ-sectional curvature c and is denoted by $\widetilde{M}^{2n+1}(c)$. As examples of Sasakian space form, $\mathbb{R}^{2n+1}$ and S^{2n+1} are equipped with Sasakian space form structures(more details in [3] and [13]). The curvature of a Sasakian space form $\widetilde{M}^{2n+1}(c)$ is given by [13]

$$\begin{aligned} \widetilde{R}(X, Y)Z = & \frac{c+3}{4}(\widetilde{g}(Y, Z)X - \widetilde{g}(X, Z)Y) \\ & + \frac{c-1}{4}(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \widetilde{g}(X, Z)\eta(Y)\xi \\ & - \widetilde{g}(Y, Z)\eta(X)\xi + \widetilde{g}(\phi Y, Z)\phi X - \widetilde{g}(\phi X, Z)\phi Y - 2\widetilde{g}(\phi X, Y)\phi Z), \end{aligned} \tag{2.5}$$

for any $X, Y, Z \in T\widetilde{M}$.

2.2. Pseudo-Parallel Legendrian Submanifolds

Let M^n be an n-dimensional submanifold of a Sasakian space form $\widetilde{M}^{2n+1}(c)$. If the one-form η constrained in M is zero, then we say M is a *Legendrian submanifold*. It is well-known that for such a submanifold ϕ maps any tangent vector to M at any $p \in M$ into the normal vector space $T_p^{\perp}M$, i.e. $\phi T_pM \subset T_p^{\perp}M$. Actually, a Legendrian submanifold is special a *C-totally real submanifold*(i.e. the unit vector field ξ is orthonormal to M) . Therefore we obtain from (2.3) and (2.4) that for any $X, Y \in TM$,

$$\widetilde{g}(\phi X, \phi Y) = g(X, Y), \ \ \eta(X) = \widetilde{g}(X, \xi) = 0,$$

where g is the induced metric of $\widetilde{g}$. As usual, ∇ and $\nabla^{\perp}$ denote by the Lev-Civita connection and normal connection on M, respectively. Then

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

where h is the second fundamental form. Similarly, the Weingarten formula is:

$$\widetilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N,$$

where N is an normal vector on M and A_N is the shape operator. The shape operator is related to the second fundamental form by

$$g(A_N X, Y) = \widetilde{g}(h(X,Y), N) = g(X, A_N Y). \tag{2.6}$$

If R and $R^\perp$ denote, respectively, the Riemannian curvature tensors corresponding to ∇ and $\nabla^\perp$, then the basic Gauss equation and Ricci equation are:

$$\begin{aligned}\widetilde{g}(\widetilde{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) + \widetilde{g}(h(X,Z),h(Y,W)) - \widetilde{g}(h(X,W),h(Y,Z)),\\ \widetilde{g}(\widetilde{R}(X,Y)N,V) &= \widetilde{g}(R^\perp(X,Y)N,V) - g([A_N,A_V]X,Y), \forall N,V \in T^\perp M.\end{aligned}$$

The Codazzi equation:

$$(\widetilde{R}(X,Y)Z)^\perp = (\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z).$$

Here $\overline{\nabla} = \nabla \bigoplus \nabla^\perp$ stands for the Van der Waerden-Bortoloti connection, given by

$$(\overline{\nabla}_X h)(Y,Z) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Moreover, the following facts are well-known:

Lemma 2.1 (see [8]). *For a Legendrian submanifold, the following equations hold:*

$$A_{\phi X} Y = A_{\phi Y} X, \tag{2.7}$$

$$A_{\phi X} Y = -\phi h(X,Y) = A_{\phi Y} X, \quad A_\xi = 0, \tag{2.8}$$

$$\widetilde{g}(h(X,Y), \phi Z) = \widetilde{g}(h(X,Z), \phi Y). \tag{2.9}$$

Therefore, it reduces from (2.2) and (2.8) that

$$\phi A_{\phi X} Y = h(X,Y) = \phi A_{\phi Y} X. \tag{2.10}$$

Moreover, using (2.10) and (2.4), from the Gauss equation we get

$$\widetilde{R}(X,Y) = R(X,Y) - [A_{\phi X}, A_{\phi Y}]. \tag{2.11}$$

For any vector fields Z, W on M, the curvature operator $\overline{R}(X,Y)$ with respect to $\overline{\nabla}$ and $X \wedge Y$ can be extended as derivations of tensor fields in usual way. Therefore

$$\begin{aligned}(\overline{R}(X,Y)\cdot h)(Z,W) =& R^\perp(X,Y)(h(Z,W))\\ &- h(R(X,Y)Z,W) - h(Z,R(X,Y)W),\end{aligned} \tag{2.12}$$

$$\begin{aligned}(X\wedge Y \cdot h)(Z,W) =& - h((X\wedge Y)Z,W) - h(Z,(X\wedge Y)W)\\ =& - g(Y,Z)h(X,W) + g(X,Z)h(Y,W)\\ &- g(Y,W)h(X,Z) + g(X,W)h(Y,Z).\end{aligned} \tag{2.13}$$

By (2.10),(2.12) and (2.13), if the normal bundle is flat, i.e. $R^\perp = 0$, then (1.1) becomes

$$\begin{aligned}-A_{\phi W}&R(X,Y)Z - A_{\phi Z}R(X,Y)W\\ &= L\big\{ - g(Y,Z)A_{\phi X}W + g(X,Z)A_{\phi Y}W\\ &\quad - g(Y,W)A_{\phi X}Z + g(X,W)A_{\phi Y}Z\big\}.\end{aligned} \tag{2.14}$$

That is, a Legendrian submanifold M^n with flat normal bundle of $\widetilde{M}^{2n+1}(c)$ is pseudo-parallel if only if the Equation (2.14) is satisfied. In particular, if $L \equiv 0$, then M is said to be *semi-parallel*. It is obvious that a totally geodesic submanifold is semi-parallel.

The following two propositions are the analogous conclusions to [5, Prop.3.1, Prop.3.2] in case of pseudo-parallel Legendrian submanifolds, respectively.

Proposition 2.2. *Let M^n be a pseudo-parallel Legendrian submanifold of Sasakian space form $\widetilde{M}^{2n+1}$. If there exists another function L' satisfies(1.1) , then $L = L'$ at least $M - V$, where $V = \{p \in M | h_p = 0\}$.*

Proof. If two functions L and L' satisfy (1.1), we have

$$(L - L')X \wedge Y \cdot h = 0.$$

Choosing an orthonormal basis $\{e_1, \cdots, e_n\}$ of T_pM, $p \in M$, we get

$$\begin{aligned}(L - L')[e_i \wedge e_j \cdot h](e_k, e_l) = &(L - L')[-h((e_i \wedge e_j)e_k, e_l) - h(e_k, (e_i \wedge e_j)e_l)] \\ =&(L - L')\{-g(e_j, e_k)h(e_i, e_l) + g(e_i, e_k)h(e_j, e_l) \\ &- g(e_j, e_l)h(e_i, e_k) + g(e_i, e_l)h(e_j, e_k)\} \\ =&(L - L')\{-\delta_{jk}h(e_i, e_l) + \delta_{ik}h(e_j, e_l) - \delta_{jl}h(e_i, e_k) + \delta_{il}h(e_j, e_k)\} = 0.\end{aligned}$$

For $i = k \neq j = l$, we have

$$(L - L')(h(e_i, e_i) - h(e_j, e_j)) = 0.$$

For $i = k = l \neq j$, we have

$$(L - L')h(e_i, e_j) = 0.$$

If $L(p) \neq L'(p)$ for $p \in M$, then

$$h(e_i, e_j) = 0, h(e_i, e_i) = h(e_j, e_j) \ \ for \, \forall i \neq j.$$

On the other hand, since when $i \neq j$,

$$\begin{aligned}&g(h(e_i, e_i), \phi e_j) = g(h(e_i, e_j), \phi e_i) = 0, \\ &g(h(e_i, e_i), \phi e_i) = g(h(e_j, e_j), \phi e_i) = g(h(e_i, e_j), \phi e_i) = 0, \\ &g(h(e_i, e_i), \xi) = 0,\end{aligned}$$

thus we obtain $g(h(e_i, e_i), N) = 0, \ i = 1, \cdots, n, \forall N \in T^\perp M$ since $\{\phi e_1, \cdots, \phi e_n, \xi\}$ is a basis of $T^\perp M$ for a Legendrian submanifold, that is, $h \equiv 0$. Consequently,

$$\{p \in M | L(p) \neq L'(p)\} \subseteq V.$$

It leads to the proposition.

Proposition 2.3. *Let M^n be a pseudo-parallel Legendrian submanifold of Sasakian space form $\widetilde{M}^{2n+1}(c)$ with flat normal bundle , then for any vector fields $X, Y \in TM$ we have*

$$R(X,Y)\phi H = L\{g(\phi H, X)Y - g(\phi H, Y)X)\}.$$

Here $H = \frac{1}{n} trh$ is the mean curvature vector.

Proof. Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame of M, Z unit vector field of T_pM for $p \in M$. For any vector U on M, using (2.6), we obtain from (2.14)

$$\begin{aligned} g(R(X,Y)Z, A_{\phi W}U) + g(R(X,Y)W, A_{\phi Z}U) \\ =& L\big\{g(Y,Z)g(A_{\phi X}W,U) - g(X,Z)g(A_{\phi Y}W,U) \\ &+ g(Y,W)g(A_{\phi X}Z,U) - g(X,W)g(A_{\phi Y}Z,U)\big\}. \end{aligned} \tag{2.15}$$

Taking $W = U = e_j$ in (2.15), we obtain

$$\begin{aligned} g(R(X,Y)Z, A_{\phi e_j}e_j) &+ g(R(X,Y)e_j, A_{\phi Z}e_j) \\ =& L\big\{g(Y,Z)g(A_{\phi X}e_j,e_j) + g(Y,e_j)g(A_{\phi X}Z,e_j) \\ &- g(X,Z)g(A_{\phi Y}e_j,e_j) - g(X,e_j)g(A_{\phi Y}Z,e_j)\big\}. \end{aligned}$$

Assume now that $\{\lambda_j\}_{j=1}^n$ are the eigenvalues of $A_{\phi Z}$ corresponding to frame $\{e_j\}_{j=1}^n$. Using (2.6) and (2.7), we get

$$\begin{aligned} -g(R(X,Y)A_{\phi e_j}e_j, Z) &+ \lambda_j g(R(X,Y)e_j, e_j) \\ =& L\big\{g(Y,Z)g(A_{\phi e_j}e_j,X) + g(Y,e_j)g(A_{\phi Z}e_j,X) \\ &- g(X,Z)g(A_{\phi e_j}e_j,Y) - g(X,e_j)g(A_{\phi Z}e_j,Y)\big\} \\ =& L\big\{g(Y,Z)g(A_{\phi e_j}e_j,X) + \lambda_j g(Y,e_j)g(e_j,X) \\ &- g(X,Z)g(A_{\phi e_j}e_j,Y) - \lambda_j g(X,e_j)g(e_j,Y)\big\}. \end{aligned}$$

i.e.

$$-g(R(X,Y)A_{\phi e_j}e_j, Z) = L\big\{g(Y,Z)g(A_{\phi e_j}e_j,X) - g(X,Z)g(A_{\phi e_j}e_j,Y)\big\}.$$

Therefore

$$\begin{aligned} g(R(X,Y)\phi H, Z) &= -\frac{1}{n}\sum_{j=1}^{n} g(R(X,Y)A_{\phi e_j}e_j, Z) \\ &= L\big\{g(Y,Z)g(\phi H,X) - g(X,Z)g(\phi H,Y)\big\}. \end{aligned}$$

It completes the proof of proposition.

3. MAIN RESULTS

Theorem 3.1. *Let M^n be a Legendrian submanifold of Sasakian space form $\widetilde{M}^{2n+1}(c)(c \leq 1)$ with flat normal bundle, then M^n is pseudo-parallel if and only if it is semi-parallel or totally geodesic.*

Proof. By the curvature tensor (2.5) of $\widetilde{M}$, for any $X, Y \in TM$ we have

$$\widetilde{R}(X,Y)\phi H = \frac{c+3}{4}(g(Y,\phi H)X - g(X,\phi H)Y).$$

Since $R^{\perp} = 0$, the Ricci equation reduces to $[A_{\phi X}, A_{\phi Y}] = 0$, which implies $\widetilde{R}(X,Y)\phi H = R(X,Y)\phi H$ in view of (2.11). So, by Proposition 2.3 we have

$$\Big(L + \frac{c+3}{4}\Big)(g(\phi H, Y)X - g(\phi H, X)Y) = 0.$$

It yields $L = -\frac{c+3}{4}$ or $H = 0$.

When $L = -\frac{c+3}{4}$, if $c = -3$, it means that $L = 0$. If $c \neq -3$, i.e. $L \neq 0$, then it is easy to get

$$-g(Y,Z)A_{\phi X}W + g(X,Z)A_{\phi Y}W - g(Y,W)A_{\phi X}Z + g(X,W)A_{\phi Y}Z = 0 \quad (3.16)$$

due to (2.14). Thus, from (3.16) and making use of analogous argument to Proposition 2.2, we have $h = 0$, that is, M is totally geodesic.

Next we assume that $L \neq -\frac{c+3}{4}$, then $H = 0$. By (2.5), for any vector X, Y, Z tangent to M,

$$\widetilde{R}(X,Y)Z = \frac{c+3}{4}(g(Y,Z)X - g(X,Z)Y. \quad (3.17)$$

By making use of (2.14) and (3.17), we have

$$\begin{aligned}(\frac{c+3}{4} - L)\big\{ - g(Y,Z)A_{\phi X}W + g(X,Z)A_{\phi Y}W \\ - g(Y,W)A_{\phi X}Z + g(X,W)A_{\phi Y}Z = 0\big\} = 0.\end{aligned} \quad (3.18)$$

If we set $X = W = e_i$, and sum over $i = 1, \cdots, n$, using that $H = 0$, we obtain $L = \frac{c+3}{4}$ or $A_{\phi Y}Z = 0$ for any Y and Z. The second case means that M is totally geodesic. Assume in following $L = \frac{c+3}{4}$. Notice that in [14] A. Yildiz et al gave an necessary condition for a minimal pseudo-parallel C-totally real submanifold to be totally geodesic is $Ln - \frac{1}{4}(n(c+3) + c - 1) \geq 0$. Therefore, in this case, M^n is also totally geodesic.

Conversely, if M is semi-parallel or totally geodesic, obviously it is trivial pseudo-parallel.

For a constant curvature manifold if its normal bundle is flat, Cartan, E. proved the following well-known fact (see [6]):

Lemma 3.2 (Cartan, E). *Let M^n be a submanifold of constant curvature space $\widetilde{M}^{n+k}(c)$, $\{\xi_\alpha\}$ local orthogonal normal vector fields, and $\{h^\alpha\}$ the second fundamental forms corresponding to $\{\xi_\alpha\}$. Then in every point of M, all the H^α are mutually diagonalizable if and only if the normal bundle of M is flat.*

By Lemma 3.2, for any $p \in M$ there exists a local orthogonal frame $\{e_i\}$ of M^n such that all the second fundamental form tensors are mutually diagonalizable, namely, for any unit normal vector field N,

$$A_N(e_i) = \lambda_i^N e_i,$$

where λ_i^N are the principle curvatures of M with respect to N.

Theorem 3.3. *For $n \geq 2$, let M^n be a Legendrian submanifold of Sasakian space form $\widetilde{M}^{2n+1}(c)$ with flat normal bundle. If M^n is pseudo-parallel, then $L = \frac{c+3}{4}$ or M is minimal.*

Proof. For a Legendrian submanifold M we may choose an orthonormal basis of $T_p^{\perp}M$ of the form $\{e_{n+1} = \phi e_1, \cdots, e_{2n} = \phi e_n, e_{2n+1} = \xi\}$. For any $i, j \in \{1, \cdots, n\}$, denote λ_i^{n+j} by the principle curvature with respect to normal vector field ϕe_j, i.e.,

$$A_{\phi e_j} e_i = \lambda_i^{n+j} e_i. \tag{3.19}$$

Thus in this case the mean curvature vector can be written as

$$H^{n+j} = \frac{1}{n} \sum_i \lambda_i^{n+j}.$$

In view of (3.18), setting $X = e_i$, $Y = e_j$, $Z = e_k$ and $W = e_l$, we have

$$\Big(\frac{c+3}{4} - L\Big)\big\{\delta_{il} A_{\phi e_j} e_k - \delta_{jk} A_{\phi e_i} e_l + \delta_{ik} A_{\phi e_j} e_l - \delta_{jl} A_{\phi e_i} e_k\big\} = 0,$$

where $g(e_i, e_j) = \delta_{ij}$ and $1 \leq i, j, k, l \leq n$. Using (3.19), we obtain

$$\Big(\frac{c+3}{4} - L\Big)\big\{\lambda_k^{n+j} \delta_{il} e_k - \lambda_l^{n+i} \delta_{jk} e_l + \lambda_l^{n+j} \delta_{ik} e_l - \lambda_k^{n+i} \delta_{jl} e_k\big\} = 0.$$

Moreover, we have

$$\Big(\frac{c+3}{4} - L\Big)\big\{\lambda_k^{n+j} \delta_{il}\delta_{ks} - \lambda_l^{n+i} \delta_{jk}\delta_{ls} + \lambda_l^{n+j} \delta_{ik}\delta_{ls} - \lambda_k^{n+i} \delta_{jl}\delta_{ks}\big\} = 0. \tag{3.20}$$

If we assume $j = s$, $i = k$ in (3.20), it reduces to

$$\Big(\frac{c+3}{4} - L\Big)\big\{\lambda_i^{n+i} \delta_{il} - \lambda_l^{n+i} \delta_{il} + \lambda_l^{n+l} - \lambda_i^{n+i} \delta_{il}\big\} = 0.$$

Further, by summing over $i = 1, \cdots, n$, we have

$$\Big(\frac{c+3}{4} - L\Big)\Big\{\sum_i \lambda_i^{n+i} \delta_{il} - \lambda_l^{n+l} + n\lambda_l^{n+l} - \sum_i \lambda_i^{n+i} \delta_{il}\Big\} = 0. \tag{3.21}$$

Because it follows from (2.7) that

$$\lambda_i^{n+l} = g(A_{\phi e_l} e_i, e_i) = g(A_{\phi e_i} e_i, e_l) = \lambda_i^{n+i} \delta_{il},$$

therefore Equation (3.21) implies

$$\Big(\frac{c+3}{4} - L\Big)\lambda_l^{n+l} = 0. \tag{3.22}$$

On the other hand, using (3.19), it follows from (2.15) that

$$\begin{aligned} -\lambda_s^{n+l} R_{ijks} - \lambda_s^{n+k} R_{ijls} = L\{ &- \delta_{jk}\delta_{ls}\lambda_l^{n+i} + \delta_{ik}\delta_{ls}\lambda_l^{n+j} \\ &- \delta_{jl}\delta_{ks}\lambda_k^{n+i} + \delta_{il}\delta_{ks}\lambda_k^{n+j}\}. \end{aligned} \tag{3.23}$$

Since

$$R_{ijks} = \frac{c+3}{4}(\delta_{jk}\delta_{is} - \delta_{ik}\delta_{js}), \quad R_{ijls} = \frac{c+3}{4}(\delta_{jl}\delta_{is} - \delta_{il}\delta_{js}), \tag{3.24}$$

by substituting (3.24) into (3.23), we get

$$\begin{aligned} &-\frac{c+3}{4}\{\lambda_s^{n+l}(\delta_{jk}\delta_{is} - \delta_{ik}\delta_{js}) + \lambda_s^{n+k}(\delta_{jl}\delta_{is} - \delta_{il}\delta_{js})\} \\ &\qquad = L\{-\delta_{jk}\delta_{ls}\lambda_l^{n+i} + \delta_{ik}\delta_{ls}\lambda_l^{n+j} - \delta_{jl}\delta_{ks}\lambda_k^{n+i} + \delta_{il}\delta_{ks}\lambda_k^{n+j}\}. \end{aligned} \tag{3.25}$$

In the same way, putting $i = k$, $j = s$ and summing over $i = 1, \cdots, n$ and $j = 1, \cdots, n$ in (3.25), respectively, we have

$$\frac{c+3}{4}H^{n+l} = L\lambda_l^{n+l}. \tag{3.26}$$

Combing (3.26) with (3.22), we concluded that if $(L - \frac{c+3}{4})H^{n+l} = 0$. This completes the proof of theorem.

It is easy to show the following corollary from (3.26):

Corollary 3.4. *For $n \geq 2$ and $c \neq -3$, let M^n be a Legendrian submanifold of Sasakian space form $\widetilde{M}^{2n+1}(c)$ with flat normal bundle. If M^n is semi-parallel then it is a minimal submanifold.*

In [14, Corollary 6], the authors showed that for a minimal Legendrian submanifold M^n of Sasakian space form $\widetilde{M}^{2n+1}(c)$, if it is semi-parallel and satisfies $n(c+3) + c - 1 \leq 0$, then it is totally geodesic. Thus by Corollary 3.4, we have the following corollary:

Corollary 3.5. *For $n \geq 2$ and $c < -3$, let M^n be a Legendrian submanifold of Sasakian space form $\widetilde{M}^{2n+1}(c)$ with flat normal bundle. If M^n is semi-parallel then it is totally geodesic.*

Note that Blair proved the following conclusion:

Theorem 3.6 ([4]). *Let M^n be a minimal C-totally real submanifold of $(2n+1)$-Sasakian space form $\widetilde{M}(c)$. Then the following are equivalent:*

1) M^n is totally geodesic,

2) M^n is of constant curvature $K = \frac{1}{4}(c+3)$,

3) $S = \frac{1}{4}(n-1)(c+3)$,

4) $\kappa = \frac{1}{4}n(n-1)(c+3)$,

where S and κ are the Ricci curvature and scalar curvature of M, respectively.

Since the normal bundle is flat, M^n is of constant curvature $K = \frac{c+3}{4}$, in view of Corollary 3.4, we have

Corollary 3.7. *Let M^n be a Legendrian submanifold of Sasakian space form $\widetilde{M}^{2n+1}(c)$ $(c \neq -3)$ with flat normal bundle. If M^n is semi-parallel the following conclusions are equivalent:*

1) M^n is totally geodesic,

2) $S = \frac{1}{4}(n-1)(c+3)$,

3) $\kappa = \frac{1}{4}n(n-1)(c+3)$.

Now recall that an non-totally geodesic Legendrian *H-umbilical submanifold* M^n of Sasakian manifold $\widetilde{M}^{2n+1}$ is a Legendrian submanifold and its second fundamental form satisfies the following forms:

$$\begin{aligned} &h(e_1, e_1) = \lambda\phi e_1,\ h(e_2, e_2) = \cdots = h(e_n, e_n) = \mu\phi e_1, \\ &h(e_1, e_j) = \mu\phi e_j,\ h(e_j, e_k) = 0,\ 2 \leq j \neq k \leq n, \end{aligned} \tag{3.27}$$

for some suitable functions λ and μ with respect to some suitable orthonormal local frame field $\{e_i\}$ of M [12].

Theorem 3.8. *Assume that M^n is a Legendrian H-umbilical submanifold of Sasakian space form $\widetilde{M}^{2n+1}(c)$ with flat normal bundle connection. If M^n is pseudo-parallel then either $L = \frac{c+3}{4}$, or $n = 1$.*

Proof. We consider $\{e_1, \cdots, e_n\}$ as (3.27), then from (3.18) we get

$$\Big(\frac{c+3}{4} - L\Big)\big\{\delta_{il}A_{\phi e_j}e_k - \delta_{jk}A_{\phi e_i}e_l + \delta_{ik}A_{\phi e_j}e_l - \delta_{jl}A_{\phi e_i}e_k\big\} = 0. \tag{3.28}$$

Assume that $j = 1$ and $i = k$ in (3.28), a straightforward calculation implies

$$\Big(\frac{c+3}{4} - L\Big)\{(n-1)(\lambda - \mu)e_1 + n\mu\sum_{l=2}^{n} e_l\} = 0. \tag{3.29}$$

If $L \neq \frac{c+3}{4}$ then the above equation implies that $\mu = 0$ and $(n-1)(\lambda - \mu) = 0$ since $\{e_i\}$ is orthonormal, that is, $n = 1$.

REFERENCES

[1] Asperti, A. C., Lobos, G. A., & Mercuri, F. (2002). Pseudo-parallel immersions in space forms. *Math. Contemp.*, *17*, 59–70.

[2] Asperti, A. C., Lobos, G. A., & Mercuri, F. (2002). Pseudo-parallel submanifolds of a space form. *Adv. Geom.*, *2*, 57–71.

[3] Blair, D. E. (2002). *Rimemannian geometry of contact and symplectic manifolds*. Birkhäuser.

[4] Blair, D. E. (1976). *Contact manifolds in riemannian geometry.* Lect Notes Math 509. Berlin Heibelberg New York: Springer.

[5] Chacón, P. M., & Lobos, G. A. (2009). Pseudo-parallel Lagrangian submanifolds in complex space forms. *Differential Geometry and Its Applications*, *27*, 137–145.

[6] Chen, B. Y. (1973). *Geometry of submanifolds.* New York: Dekker.

[7] Deszcz, R. (1992). On pseudosymmetric spaces. *Bull. Belg. Math. Soc.*, *44*, 1–34.

[8] Dillen, F., & Varancken, L. (1990). C-totally real submanifolds of Sasakian space forms. *J. Math. Pure. Appl.*, *69*, 85–93.

[9] Lobos, G. A. (2002). Pseudo-parallel surfaces in space forms (pp.197–204). In *Differential Geometry, Valencia* (2001). World Scientific Publishing, River Edger, NJ.

[10] Lobos, G. A., & Tojeiro, R. (2006). Pseudo-parallel submanifolds with flat normal bundle of space forms. *Glasg. Math. J.*, *48*, 171–177.

[11] Lobos, G. A., & Ortega, M. (2004). Pseudo-parallel real hypersurfaces in complex space forms. *Bull. Korean. Math. Soc.*, *41*, 609–618.

[12] Mihai, I., & Rădulescu, I. N. (2011). An improved Chen-Ricci inequality for Legendrian submanifolds in Sasakian space forms. *J. Adv. Math. Stud.*, *4*(2), 51–58.

[13] Yano, K., & Kon, M. (1984). *Structures on manifolds.* World Scientific: Singapore.

[14] Yildiz, A., Muranthan, C., Arslan, K., & Ezentas, R. (2007). C-totally real pseudo-parallel submanifolds of Sasakian space forms. *Monatsh. Math.*, *151*, 247–256.