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Stress Analysis Non Standard of Curved Beams

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Abstract
This article contains the results of a study aimed at examining the stress analysis of a curved beam with a rectangular cross section with a neutral-line which is contained in a plane $\boldsymbol{\pi}$. The forces that load the beam are located in the plane $\boldsymbol{\pi}$. Bending moments have pseudo-vectors perpendicular to the same plane. These kinds of loads do not stress the beam with any torsion effect. Bending moment M, shear force T and axial force N are available at each point P of the beam. Stresses σ and τ are obtained without calculating the deformed line (today, however, the Finite Element Method demands this kind of calculation). In fact, non standard stress analysis allows to calculate all the strains and stresses whilst only knowing the values of M, N, T and of the bending-radius ρ_n of the neutral-line at a given point in the neutral-line.

Key words: Elastic stability; Curved beam; Stress analysis; Non standard stress analysis

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INTRODUCTION
We consider the curve γ_G formed by the set of the centers of mass of all the cross sections of a solid curved beam. We can link the infinitesimal mass of the correspondent cross section to each centre of mass. Then γ_G can be considered as a suitable "concentration" of the mass of the continuous beam. In a similar way, we consider the neutral-line γ (defined in parametric form in accordance with the parameter t and calculated according to a pure bending moment). We have given the name of *elastic-fiber* to this kind of neutral-line. Then γ can be considered as a suitable "concentration" of stresses. So, when we consider a beam loaded without torsion effects, we are able to verify the following properties of the elastic-fiber:

The normal stress σ_{fl}, due to the bending moment $M(t)$, does not increase the length of the infinitesimal elements defined on the elastic-fiber. σ_{fl} implies only a change of the radius $\rho_n(t)$ of the elastic-fiber. Whereas the normal stress σ_{tr}, due the axial force $N(t)$, loads the infinitesimal element of the elastic-fiber and implies its deformation. We will demonstrate that the average values of the normal stresses σ_{fl} and σ_{tr} distributed over the section of the beam, coincide with the values of σ_{fl} and σ_{tr} on the elastic-fiber. Among all the fibers of the beam, only the elastic-fiber has this property. Hence, in order to plan and analyze the stress field of a curved beam, the elastic-fiber is a very good analytic model[1] of a solid beam. In the analysis of the deformation of the beam, we will ignore the effect of the shear stress τ due to the shear force $T(t)$.

1. DEFINITION OF THE PROBLEM
We define the equation of the elastic-fiber γ of the curved beam in parametric form. The beam has a longitudinal plane of symmetry and its loading is known. The cross section of the curved beam is rectangular. Its thickness is h and its width is b. Along the curve γ, h and b have a constant value. Firstly we attempt to calculate the field of

[1] The analytic model allows to associate a set of parameters obtained by analyzing the properties of a solid continuous beam to each point of a mathematic curve γ.

the normal stress $\sigma_T = \sigma_{fl} + \sigma_{tr}$[2]. Then we have:

$$\gamma : \begin{cases} X(t) \\ Y(t) \end{cases} \qquad 0 < t < t_0 \tag{1}$$

γ is the equation of a plane curve. The plane $\boldsymbol{\pi}$ that contains the neutral line γ is the same longitudinal vertical plane of symmetry as the beam. Loads applied to the curved beam are: 1) Moments whose representative pseudo-vectors are perpendicular to the plane that contains γ (the loading plane). 2) Forces situated on the plane of γ. These hypotheses on the beam and its loading, allow to reduce the analyzed problem to a simple case of bending and shear, without any effect of torsion. In order to study the stress field and deformation of the beam we will use the Linear Elasticity Hypothesis and the Principle of Superimposition of the Effects. Moreover, we will consider the beam's material to be Isotropic. We will also refer to the Hyper-Real Theory (Ref. 1).

On the curve γ we define (see Figure 1):

- a curved-coordinate $s = s(t)$
- a unitary vector $\vec{\tau} = \dfrac{\left(X^I(t), Y^I(t)\right)}{\sqrt{[X^I(t)]^2 + [Y^I(t)]^2}}$ tangential to each point $P(t)$ of the curve, and having a direction in accordance with the increasing values of "s"
- a unitary vector perpendicular to $\vec{\tau}$ and rotated anticlockwise of $\pi/2$ in respect to it. If $\vec{n}$ is orientated in accordance with the concavity of the curve γ, the bending radius is positive (see the radius ρ_B in Figure 1) and if vice versa it is negative (see the radius ρ_A in Figure 1).

The generic cross section $\Sigma(t)$ of the beam is rectangular (h is the height of the cross section and b is the width). The cross section is symmetrical in respect to the ξ-axis contained on the plane of γ and having its origin at the point $P(t)$ of the curve γ (see Figure 1 and Figure 2).

We can build the curved beam by sliding the cross section Σ perpendicularly at each point $P(t)$ of γ (see Figure 1 and Figure 2). $P(t)$ is the intersection point between Σ and γ. In order to achieve this construction, it has to be remembered that:

1) (see Figure 2). The distance $\xi_G(t)$ between $P(t)$ and the centre of mass $G(t)$ of the cross section Σ, varies in function of the parameter "t". $\xi_G(t)$ will be calculated later.

2) (See Figure 1). If we consider the cross section Σ of the beam, the unitary vector $\vec{m}$, perpendicular to the plane of Σ, must be parallel to the unitary vector $\vec{\tau}$ tangent to γ in $P(t)$. The radius $\rho_n(t)$ of the elastic-fiber γ in $P(t)$, is situated on the ξ-axis. We can obtain the ξ-axis as the intersection of the plane of γ with the plane of the cross section Σ. The sign of ρ_n is in accordance with increasing ξ.

3) Both the thickness h and the width b of the cross section Σ (see Figure 2), are constant.

4) When we construct the beam by sliding the cross section Σ along the curve γ in $P(t)$, we must be careful that we never generate any point of singularity or any overlapping of the material. Then we have to verify the conditions:

$$\begin{aligned} &\rho_n(t) > 0 \Rightarrow |\rho_n(t)| > |c(t)| \\ &\rho_n(t) < 0 \Rightarrow |\rho_n(t)| > |-a(t)| \end{aligned} \tag{2}$$

We can notice that Σ is always perpendicular to the curve γ_G [generated by the motion of the point $G(t)$ of the centre of mass of the cross section $\Sigma(t)$ when it slides γ in $P(t)$]. Then the curved beam can be more easily constructed by perpendicularly sliding the centre of mass of the cross section $\Sigma(t)$ along the curve γ_G. The equation of γ_G will be calculated later from the equation of γ.

Furthermore, we see that the normal stress on the elastic-fiber [that is the neutral line generated by loading the beam with a pure bending moment $M(t)$] is always zero:

$$\sigma_{fl}(\xi = 0) = 0 \tag{3}$$

Whereas the normal stress due to the Normal Force $N(t)$, will later be demonstrated to be:

$$\sigma_{tr}(\xi = 0) = \frac{N(t)}{bh} \tag{4}$$

These results are of interest because they verify that the stresses σ_{fl}, σ_{tr} in correspondence to the elastic-fiber, coincide with the average values of the same stresses, calculated on the complete cross section of the beam.

The shear stresses are also non zero at $\xi = 0$. But in order to calculate the deforming-line γ_D, we ignore their effects. We assume the strain field into the infinitesimal element dV to be linear (see Figure 3). Then the calculation of $\sigma_T = \sigma_{fl} + \sigma_{tr}$ and τ, on the complete curved beam will be rigorous only in case of rigid-bodies, where $E, G \to \infty$. In fact only under this condition are all the infinitesimal strains into the curved beam linear as all the infinitesimal strains greater than the infinitesimal of the first order are negligible. Since in a rigid-beam the deforming-line γ_D coincides with the neutral line γ, we will be able to calculate the stress and strain fields into each curved beam under the condition of non-deformation[3].

Bending moment $M(t)$, normal force $N(t)$ and shear force $T(t)$, must be calculated by applying the loads on the elastic-fiber γ (Figure 4).

We can define the stress field in a complete way because the problem is symmetric. In an infinitesimal element of the beam we have (see Figure 1, Figure 2 and Figure 5 to define the coordinate s and the axis ξ, ζ):

[2] The normal stresses due to the contemporary action of the normal force and the bending moment are calculated according to the loads applied on the beam. The shear stress due to the shear force will be considered later.
[3] Cross sections $\Sigma(t)$ are always plane and perpendicular also in the deformed-line γ_D. In fact, the infinitesimal strains are always linear. Then our calculations are only valid when discussing the rigid-beams and the elastic-beams under the non deformed conditions.

$$\begin{aligned}&\tau_{s\zeta} = \tau_{\zeta s} = 0\\ &\tau_{\xi\zeta} = \tau_{\zeta\xi} = 0\\ &\sigma_{\zeta} = \sigma_{\xi} = 0\\ &\sigma_s = \sigma_T(\xi, t)\\ &\tau_{s\xi} = \tau_{\xi s} = \tau(\xi, t)\end{aligned} \tag{5}$$

If we consider a fiber of the beam (the gap between two adjacent fibers is δ (see Figure 6), where δ is the smallest positive real number[4]), we can notice that $\tau(\xi + \delta, t) = -\tau^I(\xi, t)$ (see Figure 6). Then they vanish and do not contribute to the stretching or to the shortening of the fiber of the beam in the direction of the curved coordinate "*s*".

We will verify that $\sigma_T \tau$ only depend on the geometry and the loads applied to the beam. Stresses do not depend on the material of the beam[5].

2. THE ELASTIC-FIBER POSITION INTO THE CURVED BEAM WHEN THE BENDING MOMENT *M*(*t*) AND THE NORMAL FORCE *N*(*t*) ARE APPLIED TOGETHER

We consider the analytic model γ of a curved beam, when the bending moment $M(t)$ and the normal force $N(t)$ are applied together at the point $P(t)$ (Figure 4). The position of the elastic-fiber coincides with the neutral line position and it is only a function of the geometry of the curved beam. The position of the elastic-fiber is obtained by the values of $a(t)$ and $c(t)$ in the cross section (see Figure 2). The functions $a(t)$ and $c(t)$ [being $a(t) + c(t) = h$] will be calculated later.

In our analysis we will use the following geometric relations (see Figure 3):

$$ds = \sqrt{[X^I(t)]^2 + [Y^I(t)]^2}\,dt \tag{6}$$

$$ds = \rho_n d\vartheta \tag{7}$$

$$\rho_n = \frac{\left\{\left[X^I(t)\right]^2 + \left[Y^I(t)\right]^2\right\}^{3/2}}{\begin{vmatrix} X^I(t) & Y^I(t)\\ X^{II}(t) & Y^{II}(t)\end{vmatrix}} \tag{8}$$

The infinitesimal element of Figure 3 differs from the exact element for infinitesimals of a higher order than the first. Then, after the integration, some formulae with infinitesimal errors (negligible in the real field) are obtained.

When we apply the bending moment $M(t)$ and the normal force $N(t)$ to the infinitesimal element ABCD it changes its form into ABFE. The normal force will deform the elastic-fiber PP' for an amount Δ_0.

We have:

$$PP' = ds = \rho_n d\vartheta + \circ(d\vartheta) \tag{9}$$

with $\circ(d\vartheta)$ we have indicated an infinitesimal amount of a higher order than $d\vartheta$.

The triangles OPP' and P'RR' are similar. Then:

$$RR' = PP'\frac{\xi}{\rho_n} = \frac{\xi}{\rho_n}ds \tag{10}$$

On the other hand, we have:

$$\begin{aligned}d\varphi &= \frac{ds + \Delta_0}{\rho_D} = \frac{R'R''}{TO' - \xi}\\ &= \frac{R'R''}{P'Q - \xi} \Leftrightarrow R'R'' = \left(\frac{ds + \Delta_0}{\rho_D}\right)\left(P'Q - \xi\right)\end{aligned} \tag{11}$$

being:

$$TO' = \frac{\Delta_0}{d\varphi} = P'Q = \frac{\rho_D \Delta_0}{ds + \Delta_0} \tag{12}$$

Then, by Equation (10), Equation (11) and Equation (12), we can calculate the stretching:

$$\begin{aligned}\Delta(\xi, t) &= RR'' = RR' + R'R''\\ &= \frac{\xi}{\rho_n}ds + \left(\frac{ds + \Delta_0}{\rho_D}\right)\left(\frac{\Delta_0 \rho_D}{ds + \Delta_0} - \xi\right)\end{aligned} \tag{13}$$

of a fiber that is distant by an amount of $\xi > 0$ from the elastic-fiber PP'. Then the strain is:

$$\varepsilon(\xi, t) = \frac{RR''}{SR} = \frac{RR''}{SR' - RR'} = \frac{RR''}{PP' - RR'} \tag{14}$$

By substituting Equation (9), Equation (10) and Equation (13) into Equation (14) , we obtain:

$$\varepsilon(\xi, t) = \frac{\xi(\rho_D - \rho_n)}{\rho_D(\rho_n - \xi)} + \frac{\rho_n(\rho_D - \xi)}{\rho_D(\rho_n - \xi)}\frac{\Delta_0}{ds} \tag{15}$$

ε_0 is the strain of the elastic fiber PP'. We have:

$$\varepsilon_0 = \frac{\Delta_0}{ds} \tag{16}$$

we will obtain:

$$\varepsilon(\xi, t) = \frac{\xi(\rho_D - \rho_n) + \rho_n \varepsilon_0(\rho_D - \xi)}{\rho_D(\rho_n - \xi)} \tag{17}$$

The correspondent stress will be (in the elastic field):

$$\sigma_T(\xi, t) = \frac{E[\xi(\rho_D - \rho_n) + \rho_n \varepsilon_0(\rho_D - \xi)]}{\rho_D(\rho_n - \xi)} \tag{18}$$

The limits of the cross section are (see Figure 2):

$$\begin{aligned}&\xi = -a(t)\\ &\xi = c(t)\end{aligned} \tag{19}$$

They are calculated by a mathematic system formed by the integral:

$$b\int_{-a(t)}^{c(t)} \sigma_{fl}(\xi, t)d\xi = 0 \tag{20}$$

[Equation (20) is the condition of equilibrium of the

[4] The meaning of "the smallest positive real number" is considered in Ref. 1.

[5] Since the proposed stress analysis considers in a complete and coherent way the cases of the rigid beams and the flexible beams in the non deformed conditions, we trust this could be a good theory for beams with either a low or high buckling.

infinitesimal element, loaded by the bending moment $M(t)$, with respect to a straight movement along the increasing $s(t)$] and under the condition:

$$a(t) + c(t) = h \tag{21}$$

with: h = thickness of the curved beam.

Into Equation (20) we need to introduce the function $\sigma_{fl}(\xi,t)$ that we obtain by putting $\varepsilon_0 = 0$ into Equation (18). Actually for a pure bending moment, ε_0 must be zero.

$$\begin{cases} a(t)+c(t)=h \\ \rho_n(t)\log\left(\dfrac{\rho_n(t)+a(t)}{\rho_n(t)-c(t)}\right)=h \end{cases} \tag{22}$$

Developing the calculations, we obtain:

$$a(t)=\frac{h\exp\left(\dfrac{h}{\rho_n(t)}\right)}{\exp\left(\dfrac{h}{\rho_n(t)}\right)-1}-\rho_n(t) \tag{23}$$

$$c(t)=\rho_n(t)-\frac{h}{\exp\left(\dfrac{h}{\rho_n(t)}\right)-1} \tag{24}$$

Imposing the equilibrium of the infinitesimal element of Figure 3 with respect to the straight movement in the direction "s", we can obtain:

$$\int_{-a}^{c} b\sigma_T(\xi,t)d\xi = N(t) \tag{25}$$

On the other hand, we can impose the equilibrium for the rotation of the same infinitesimal element, by the integral:

$$\int_{-a}^{c} b\xi\sigma_T(\xi,t)d\xi + M(t) = 0 \tag{26}$$

We can substitute σ_T [obtained from Equation (18)] into Equation (25) and Equation (26), and calculate the previous integrals on the interval $[-a(t), c(t)]$ [where $a(t)$, $c(t)$ are given by Equation (23) and Equation (24)]. We obtain:

$$\frac{Eb}{\rho_D}\left\{\begin{array}{l}(\varepsilon_0+1)\rho_n(\rho_n-\rho_D)\log\left(\dfrac{\rho_n+a-h}{\rho_n+a}\right)\\ +h\left[(\varepsilon_0+1)\rho_n-\rho_D\right]\end{array}\right\}=N(t) \tag{27}$$

$$\frac{bE}{\rho_D}\left\{\begin{array}{l}(\varepsilon_0+1)(\rho_n-\rho_D)\rho_n^2\log\left(\dfrac{\rho_n+a-h}{\rho_n+a}\right)\\ -\dfrac{h}{2}\left\{\begin{array}{l}2a\left[(\varepsilon_0+1)\rho_n-\rho_D\right]\\ -\varepsilon_0\rho_n\left[h+2(\rho_n-\rho_D)\right]\\ -(h+2\rho_n)(\rho_n-\rho_D)\end{array}\right\}\end{array}\right\}=-M(t) \tag{28}$$

Since, from Equation (22) :

$$\log\left(\frac{\rho_n+a-h}{\rho_n+a}\right)=-\frac{h}{\rho_n} \tag{29}$$

Substituting Equation (29) into Equation (27) we obtain:

$$\varepsilon_0=\frac{N(t)}{Ebh} \tag{30}$$

This result demonstrates the perfect equivalence between the normal stress generated in $P(t)$ on the elastic-fiber by the shear force $N(t)$ and the average value:

$$\sigma_{tr}(0,t)=\frac{N(t)}{bh} \tag{31}$$

On the other hand, if we substitute Equation (29) and Equation (30) into Equation (28), we obtain:

$$\rho_D=\frac{\rho_n\left(Ebh+N(t)\right)\left(a(t)-\dfrac{h}{2}\right)}{M(t)+Ebh\left(a(t)-\dfrac{h}{2}\right)} \tag{32}$$

If we substitute Equation (30) and Equation (32) into Equation (18), we obtain:

$$\sigma_T(\xi,t)=\frac{-\xi M(t)+\rho_n N(t)\left(a(t)-\dfrac{h}{2}\right)}{bh(\rho_n-\xi)\left(a(t)-\dfrac{h}{2}\right)} \tag{33}$$

$a(t)$ is given from Equation (23).

From Equation (33) we obtain:

$$\begin{aligned}\sigma_T(\xi,t)&=\sigma_{fl}(\xi,t)+\sigma_{tr}(\xi,t)\\ &=\frac{-\xi M(t)}{bh(\rho_n-\xi)\left(a(t)-\dfrac{h}{2}\right)}+\frac{\rho_n N(t)}{bh(\rho_n-\xi)}\end{aligned} \tag{34}$$

If we substitute the functions:

$$\begin{aligned}\sigma_{tr}(\xi,t)&=\frac{\rho_n N(t)}{bh(\rho_n-\xi)}\\ \sigma_{fl}(\xi,t)&=\frac{-\xi M(t)}{bh(\rho_n-\xi)\left(a(t)-\dfrac{h}{2}\right)}\end{aligned} \tag{35}$$

into the following integrals:

$$\begin{aligned}&\int_{-a(t)}^{c(t)} b\xi\sigma_{tr}(\xi,t)d\xi=0\\ &\int_{-a(t)}^{c(t)} b\xi\sigma_{fl}(\xi,t)d\xi+M(t)=0\end{aligned} \tag{36}$$

$$\begin{aligned}&\int_{-a(t)}^{c(t)} b\sigma_{tr}(\xi,t)d\xi=N(t)\\ &\int_{-a(t)}^{c(t)} b\sigma_{fl}(\xi,t)d\xi=0\end{aligned} \tag{37}$$

we obtain four identities. From Equation (35) we can observe that $\lim_{\rho_n\to\infty}\sigma_{tr}(\xi,t)=\dfrac{N(t)}{bh}$. This result coincides with the behavior of a straight beam loaded by a normal traction force. Then the infinitesimal element of the elastic-fiber loaded by the normal force $N(t)$ has a behavior which is coherent with Hook's Law:

$$\Delta l = \frac{Nl}{EA} \Leftrightarrow \sigma_{tr} = \frac{N}{A} = \frac{N}{bh} \tag{38}$$

Equation (33) does not depend on Young's Modulus E, but only on the geometry of the beam and the applied loads.

The curve γ_G generated by the centre of mass of the cross sections will have the following equation:

$$\begin{aligned}\left(X_G(t), Y_G(t)\right) &= \left(X(t), Y(t)\right) - \left(a(t) - \frac{h}{2}\right)\bar{n} \\ &= \left(X(t), Y(t)\right) - \left(a(t) - \frac{h}{2}\right)\frac{\left(-Y^I(t), X^I(t)\right)}{\sqrt{\left[X^I(t)\right]^2 + \left[Y^I(t)\right]^2}}\end{aligned} \tag{39}$$

3. CALCULATION OF SHEAR STRESS GENERATED BY SHEAR FORCE *T*(*t*)

In order to study the $\tau(\xi, t)$ we have to think about the infinitesimal element shown in Figure 7. Here, we have considered the infinitesimal dx (that is the differential along the $s(t)$ direction, of the element ABCD shown in Figure 3). dx is linked to the coordinate ξ (of course for $\xi = 0$ then $dx = ds$).

In order to obtain a mathematic description of dx (see Figure 7), we have to refer to Figure 3. In it, for $\xi > 0$, dx corresponds to $SR = SR' - RR'$. If we remember Equation (6) and Equation (7), then for each real ξ, we obtain:

$$\begin{aligned}\text{SR } &= dx = ds - \xi d\vartheta = (\rho_n - \xi)d\vartheta \\ &= \frac{(\rho_n - \xi)}{\rho_n}ds = \sqrt{[X^I(t)]^2 + [Y^I(t)]^2}\,\frac{(\rho_n - \xi)}{\rho_n}dt\end{aligned} \tag{40}$$

being: $RR' = \xi d\vartheta$ and $SR' = PP' = ds$

Since the function $\tau(\xi, t)$ is unknown, we assume the following correlations to be correct:

$$\tau^I - \tau = \frac{\partial \tau}{\partial \xi}d\xi \tag{41}$$

$$\int_{-a(t)}^{c(t)} b\tau(\xi, t)d\xi = T(t) \tag{42}$$

$$\sigma_T^I - \sigma_T = \frac{\partial \sigma_T}{\partial x}dx \tag{43}$$

Equation (41) and Equation (43) verify the coherence between the positive directions defined for $M(t)$, $N(t)$ and $T(t)$, and the positive direction of the reference systems used in the mathematic simulation of the curved beam.

We can consider the following deductions to be exact:

1) When we impose the equilibrium for the rotation to the infinitesimal element shown in Figure 7 (having taken into account the properties of the hyper-real number), the real quantities:

$$\tau^I(\xi, t) \equiv \tau(\xi, t) \tag{44}$$

are coincident. Then we can write:

$$\{[b(d\xi)]\tau(\xi, t)\}dx = \{[b(dx)]\tau(\xi, t)\}d\xi \tag{45}$$

Of course, in the real field this correlation is always verified.

2) When we impose the equilibrium to the straight movement along the ξ-axis to the infinitesimal element shown in Figure 7, we obtain:

$$[b(d\xi)]\tau(\xi, t) = [b(d\xi)]\tau^I(\xi, t) \tag{46}$$

Thanks to Equation (44), this correlation is always verified.

3) When we impose the equilibrium to the straight movement along the x-axis to the infinitesimal element shown in Figure 7, we obtain:

$$(\tau^I - \tau)b(dx) + (\sigma_T^I - \sigma_T)b(d\xi) = 0 \tag{47}$$

By bearing in mind Equation (41) and Equation (43), we can deduce:

$$\frac{\partial \tau}{\partial \xi} = -\frac{\partial \sigma_T}{\partial x} \text{ with: } x = x(t, \xi) \tag{48}$$

So, from Equation (40), we obtain:

$$\frac{\partial \tau}{\partial \xi} = -\frac{\rho_n(t)}{(\rho_n(t) - \xi)}\frac{1}{\sqrt{[X^I(t)]^2 + [Y^I(t)]^2}}\frac{\partial \sigma_T}{\partial t} \tag{49}$$

We can rewrite Equation (33) as:

$$\sigma_T(\xi, t) = \frac{1}{[\rho_n(t) - \xi]}\left\{A(t) + B(t)\xi\right\} \tag{50}$$

with:

$$A(t) = \left\{\frac{\rho_n(t)N(t)}{bh}\right\} \tag{51}$$

and:

$$B(t) = \left\{\frac{-M(t)}{bh\left(a(t) - \frac{h}{2}\right)}\right\} \tag{52}$$

We can derive the Equation (50) obtaining:

$$\begin{aligned}\frac{\partial \sigma_T(\xi, t)}{\partial t} &= \frac{-\rho_n^I(t)}{[\rho_n(t) - \xi]^2}\left\{A(t) + B(t)\xi\right\} \\ &\quad + \frac{1}{[\rho_n(t) - \xi]}\left\{A^I(t) + \xi B^I(t)\right\}\end{aligned} \tag{53}$$

with:

$$A^I(t) = \frac{1}{bh}\left[\rho_n^I(t)N(t) + \rho_n(t)N^I(t)\right] \tag{54}$$

$$\begin{aligned}\rho_n^I(t) &= \frac{3\sqrt{[X^I(t)]^2 + [Y^I(t)]^2}\left\{X^I(t)X^{II}(t) + Y^I(t)Y^{II}(t)\right\}}{\left\{X^I(t)Y^{II}(t) - Y^I(t)X^{II}(t)\right\}} \\ &\quad - \frac{\left\{[X^I(t)]^2 + [Y^I(t)]^2\right\}^{3/2}\left\{X^I(t)Y^{III}(t) - Y^I(t)X^{III}(t)\right\}}{\left\{X^I(t)Y^{II}(t) - Y^I(t)X^{II}(t)\right\}^2}\end{aligned} \tag{55}$$

$$B^I(t) = \frac{1}{bh}\left[\frac{-M^I(t)\left(a(t) - \frac{h}{2}\right) + M(t)a^I(t)}{\left(a(t) - \frac{h}{2}\right)^2}\right] \tag{56}$$

By deriving Equation (23) we obtain:

$$a^I(t) = \rho_n^I(t)\left\{\left(\frac{h}{\rho_n}\right)^2 \frac{\exp\left(\frac{h}{\rho_n}\right)}{\left[\exp\left(\frac{h}{\rho_n}\right)-1\right]^2} - 1\right\} \tag{57}$$

From Equation (49) and from Equation (53) we have:

$$\frac{\partial\tau}{\partial\xi} = -\frac{\rho_n(t)}{(\rho_n(t)-\xi)}\frac{1}{\sqrt{[X^I(t)]^2+[Y^I(t)]^2}}\left\{\frac{-\rho_n^I}{[\rho_n(t)-\xi]^2}\{A(t)+B(t)\xi\} + \frac{1}{[\rho_n(t)-\xi]}\{A^I(t)+\xi B^I(t)\}\right\} \tag{58}$$

Equation (58) can be rewritten as:

$$\frac{\partial\tau}{\partial\xi} = D(t)\left\{\frac{A^I(t)+\xi B^I(t)}{(\rho_n(t)-\xi)^2} - \frac{A(t)+\xi B(t)}{(\rho_n(t)-\xi)^3}\rho_n^I(t)\right\} \tag{59}$$

with:

$$D(t) = -\frac{\rho_n(t)}{\sqrt{[X^I(t)]^2+[Y^I(t)]^2}} \tag{60}$$

Equation (59) can be integrated with respect to ξ:

$$\tau(\xi,t) = D(t)\left\{B^I(t)\log(\rho_n-\xi) - \frac{2\xi\left[B(t)\rho_n^I + A^I(t) + B^I(t)\rho_n\right] + A(t)\rho_n^I - \rho_n\left[B(t)\rho_n^I + 2\left(A^I(t)+B^I(t)\rho_n\right)\right]}{2(\rho_n-\xi)^2}\right\} + k \tag{61}$$

In order to obtain the integration constant k, we can impose the condition (42) and obtain:

$$k = -\frac{D(t)\left[B(t)\rho_n^I(t) + A^I(t) + 2\rho_n B^I(t)\right]}{\rho_n} - \frac{D(t)B^I(t)[a\log(\rho_n+a) + c\log(\rho_n-c)]}{h} + \frac{D(t)\rho_n^I(t)\left[A(t)+B(t)\rho_n\right]}{2(\rho_n+a)(\rho_n-c)} + B^I(t)D(t) + \frac{T(t)}{bh} \tag{62}$$

Through these mathematic operations, we have obtained the function $\tau(\xi, t)$. Previous results are applicable in the stress analysis of both the non deformed flexible beam and the rigid beam.

CONCLUSIONS

The proposed stress analysis can be extended to the case of non rectangular yet symmetric, cross sections. We will obtain the following results:

For each form of the cross section of the beam, the value of the strain ε_0 in correspondence to the neutral line is always valid:

$$\varepsilon_0 = \varepsilon(\xi=0,t) = \frac{N(t)}{EA} \tag{63}$$

with A = area of the symmetric cross section.

For each form of the cross section of the beam, the radius ρ_D at the point $Q(t)$ of the deformed-line [$Q(t)$ corresponds on the deformed line to the point $P(t)$ on the elastic-fiber] is always valid:

$$\rho_D(t) = \rho_n(t)\left\{\frac{\left(N(t)+EA\right)\left[a(t)-a_\infty\right]}{M(t)+EA\left[a(t)-a_\infty\right]}\right\} \tag{64}$$

with a_∞ = value of $a(t)$ when $\rho_n \to \infty$ (in this case the elastic fiber coincides with the curve ρ_G formed by the centers of mass of the cross section whilst $P(t)$ slides on the elastic fiber γ).

Beams having cross sections of a different form, but with the same area A, have the same normal stresses $\sigma_{tr}(\xi, t)$ and $\sigma_{fl}(\xi, t)$:

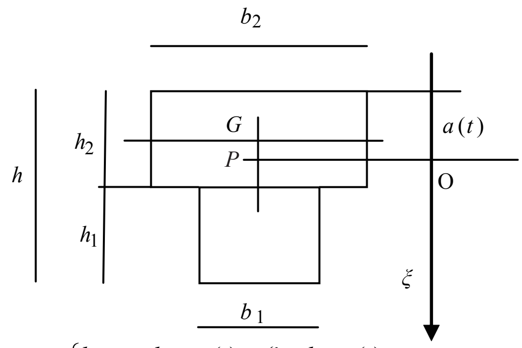
$$\sigma_{tr}(\xi,t) = \frac{\rho_n(t)N(t)}{A\left(\rho_n(t)-\xi\right)} \tag{65}$$

$$\sigma_{fl}(\xi,t) = \frac{-\xi M(t)}{A\left(\rho_n(t)-\xi\right)\left[a(t)-a_\infty\right]} \tag{66}$$

Since the Equation (65) and Equation (66) do not depend on Young's Modulus E, then the shear stress generated by the shear force, does not depend on the Torsion Modulus G. Actually the relationship $\frac{\partial\tau(\xi,t)}{\partial\xi} = -\frac{\partial\sigma_T(\xi,t)}{\partial x}$ is always true. Then the normal and shear stresses are only functions of the geometry of the beam and the applied loads. On the other hand, we can assume (without error) that the beam is a rigid beam (E, $G \to \infty$). But the strain of a rigid beam is an infinitesimal of the first order. Since the infinitesimals of an order higher than the first are infinitesimals in respect to the infinitesimals of the first order, these kinds of strain (infinitesimals) have to be strictly linear. Then, in a rigid beam, the cross sections are always plane. This reasoning implies that the proposed stress analysis allows to calculate the stress of a beam under the non-deformed conditions. When the loading conditions do not generate any torsion effect, then a beam under non-deformed conditions can be considered to be a rigid beam. Actually both kinds of beams have the same elastic-fiber. Moreover, the stresses are only calculated in accordance with the beam geometry and the applied loads. In fact the elastic characteristics of the beam's material (E, G) have no influence on the stress analysis.

We analyze three forms of cross section:

The *T* Cross Section



$$b(\xi) = \begin{cases} b_1 : & h_2 - a(t) < \xi < h - a(t) \\ b_2 : & -a(t) < \xi < h_2 - a(t) \end{cases}$$

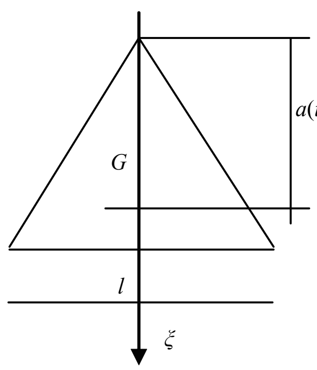
$$A = b_1 h_1 + b_2 h_2$$

$$a_\infty = \frac{b_1 h_1 (h_1 + 2h_2) + b_2 h_2^2}{2\left(b_1 h_1 + b_2 h_2\right)}$$

We obtain $a(t)$ by the solution of the transcendent equation:

$$\rho_n \left[b_2 \log\left\{ \frac{\rho_n + a(t)}{\rho_n + a(t) - h_2} \right\} + b_1 \log\left\{ \frac{\rho_n + a(t) - h_2}{\rho_n + a(t) - h} \right\} \right] = A$$

Triangular Cross Section



$$b(\xi) = l - \frac{2}{\sqrt{3}}\left(a(t) + \xi\right)$$

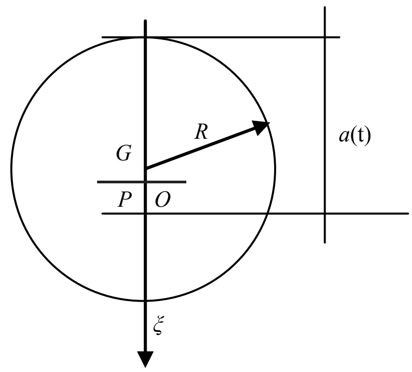
$$A = \frac{l^2\sqrt{3}}{4}$$

$$a_\infty = \frac{l}{2\sqrt{3}}$$

We obtain $a(t)$ by the solution of the transcendent equation:

$$8\rho_n \left[\rho_n - \frac{l\sqrt{3}}{2} + a(t) \right] \log\left\{ \frac{\rho_n - \frac{l\sqrt{3}}{2} + a(t)}{\rho_n + a(t)} \right\} = l\sqrt{3}\left[l\sqrt{3} - 4\rho_n \right]$$

Circular Cross Section



$$b(\xi) = 2\sqrt{\left(\xi + a(t)\right)\left(2R - \xi - a(t)\right)}$$

$$A = \pi R^2$$

$$a_\infty = R$$

$$a(t) = R\left(1 + \frac{R}{4\rho_n}\right) \quad \text{con:} \quad \rho_n > \frac{R}{2}$$

Then for a wire, the bending radius must be greater than half the radius R of the circular cross section. This condition verifies that is impossible to obtain an edge with a radius smaller than $R/2$ without deforming the material of the cross section.

REFERENCES

[1] Keisler, H. J. (1982). *Elementi di Analisi Matematica*. Padova (Italy): Piccin Editore.

APPENDIX

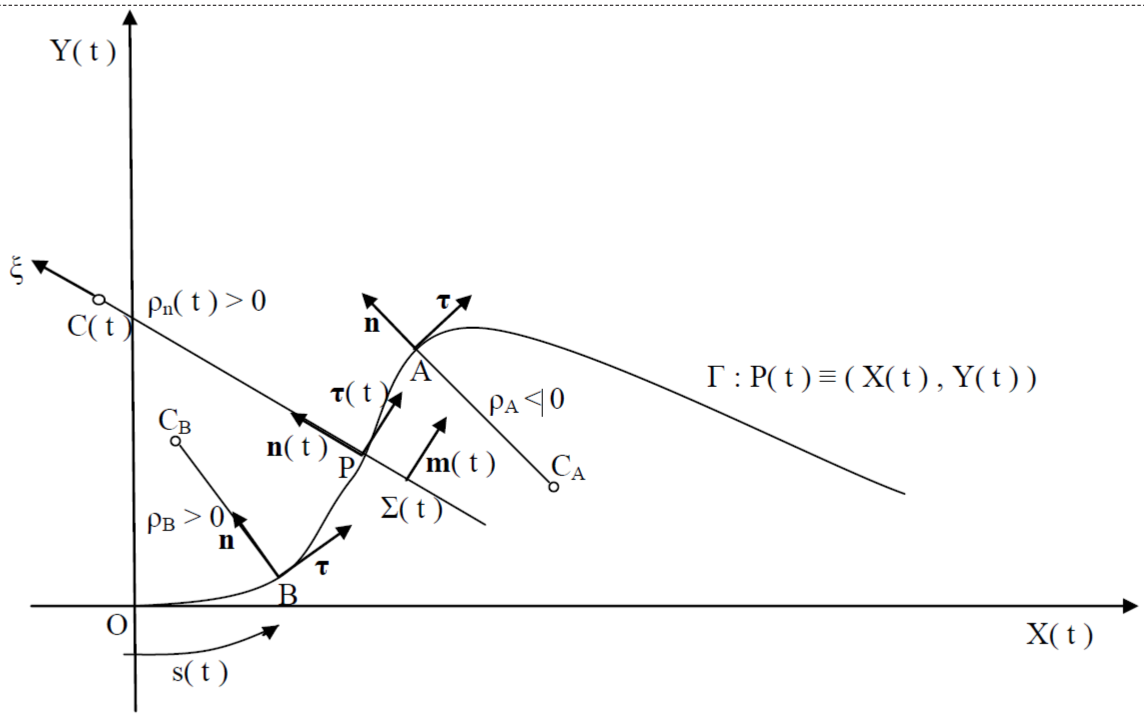


Figure 1
Geometric Convention in Studying a Curved Beam

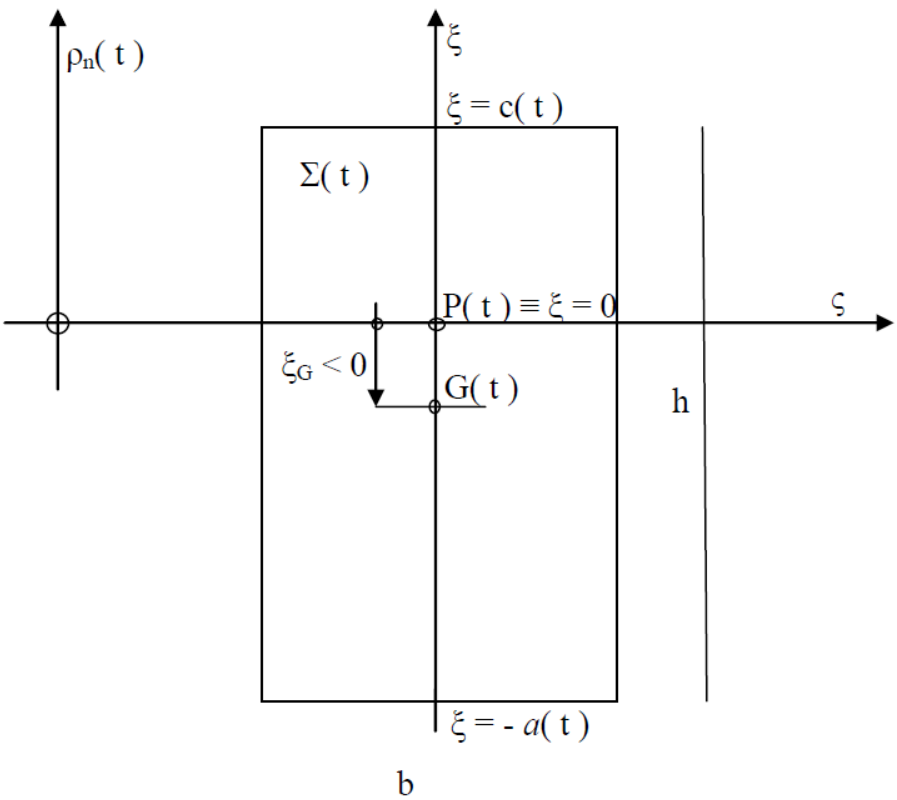
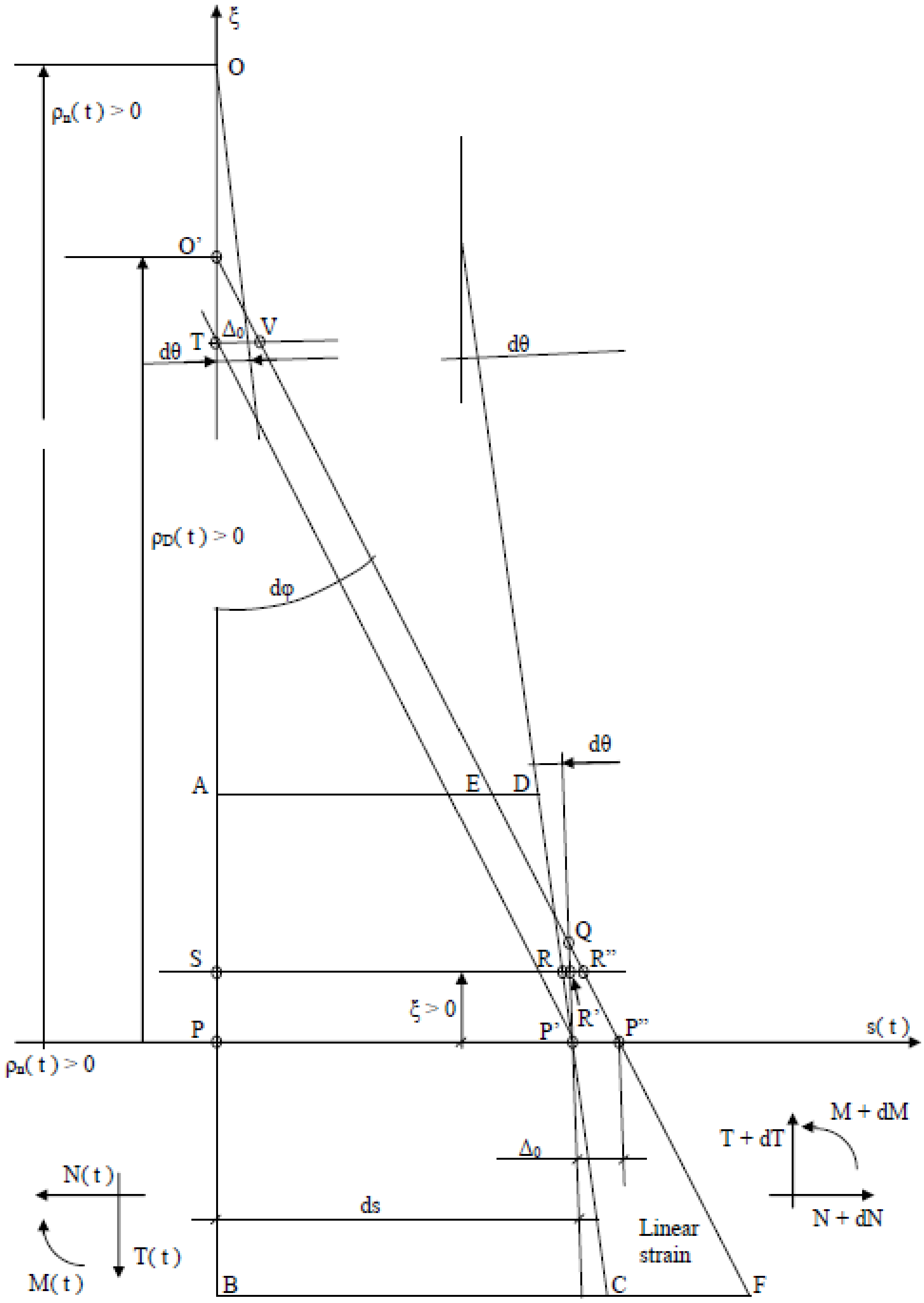


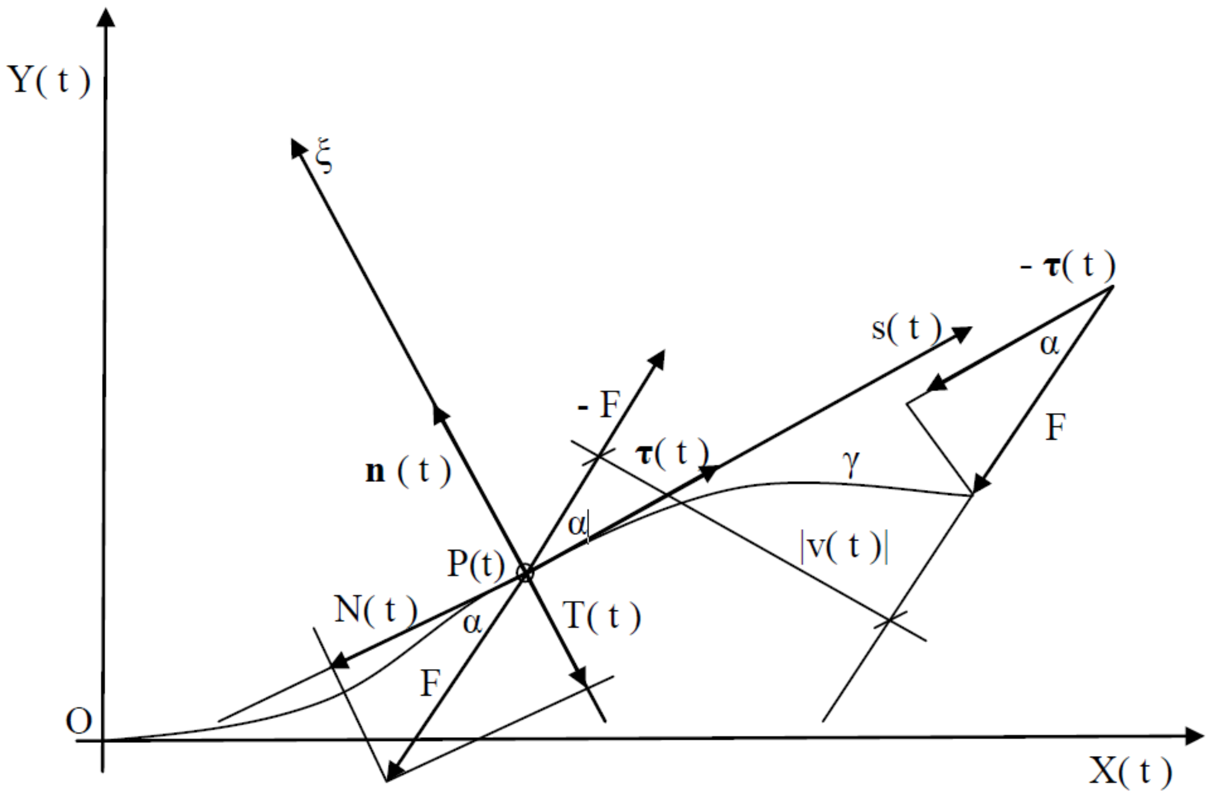
Figure 2
Geometric Conventions on the Cross Section of a Curved Beam



PÔP' = RP'Q = dθ; TP'//P''O'; TO' = P'Q = TV/dφ; P'P'' = TV = Δ_0; $\Delta_0 = \rho_D d\varphi - \rho_n d\theta$

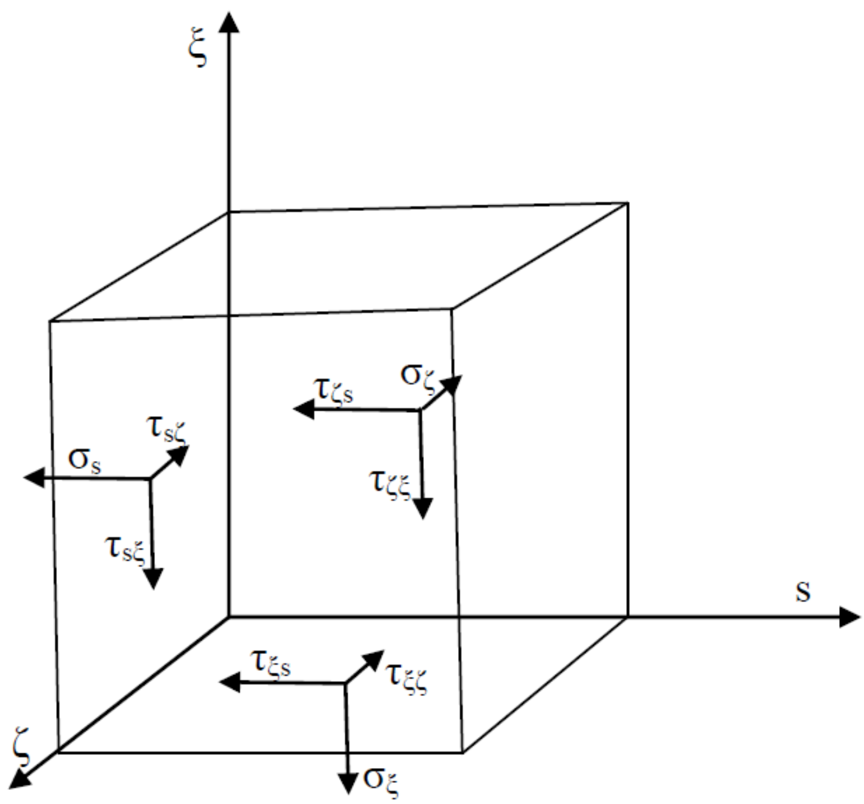
The directions pointed out in Figure 3 have to be considered as positive directions

Figure 3
Infinitesimal Element that Differs from the Exact Element only for Higher Order Infinitesimals



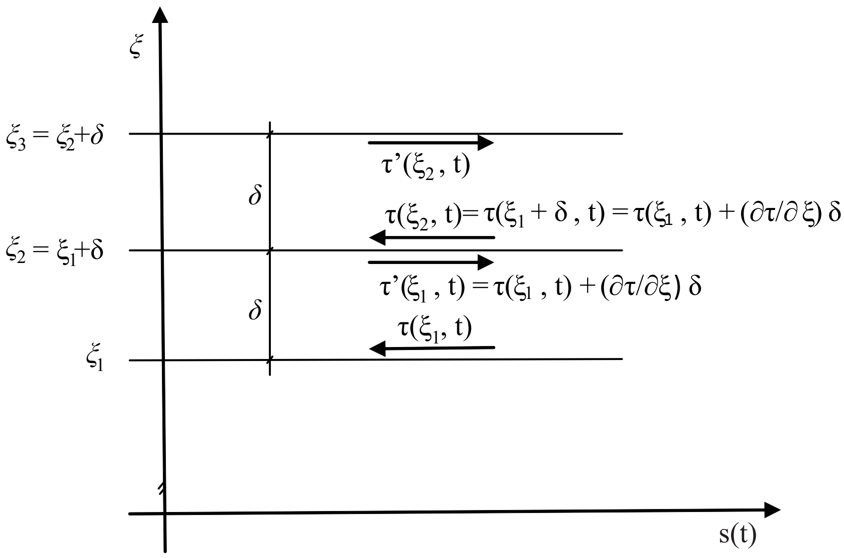
$M(t) = -F|v(t)|$; $N(t) = -F\cos(\alpha)$; $T(t) = -F\sin(\alpha)$; $\cos(\alpha) = -(\boldsymbol{F}\,\boldsymbol{\tau})/|\boldsymbol{F}|$

Figure 4
Calculation of the Loading Field: *M*(*t*), *N*(*t*), *T*(*t*)



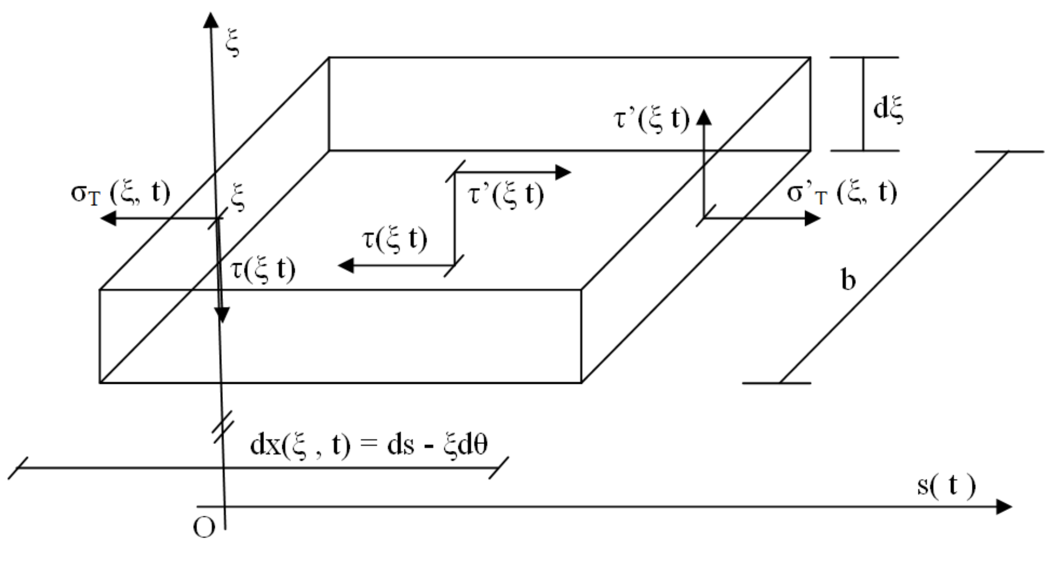
$\sigma_s = \sigma_T(\xi, t)$
$\tau_{s\xi} = \tau(\xi, t) = \tau_{\xi s}$

Figure 5
Elementary Stress in an Infinitesimal Volume of the Beam



δ is the smallest positive real number (Ref. 1). All the infinitesimals are between 0 and δ.

Figure 6
Elementary Shear Stresses in a Cross Section of a Curved Beam



$dx = ds - \xi\, d\theta = [(\rho_n - \xi)/\rho_n]ds = (\rho_n - \xi)d\theta$

Figure 7
Equilibrium of the Stresses in the Infinitesimal Element

