CSCanada
International Business and Management
Vol. 9, No. 1, 2014, pp. 14-19
DOI:10.3968/5371

ISSN 1923-841X [Print]
ISSN 1923-8428 [Online]
www.cscanada.net
www.cscanada.org

Variable Annuities and Embedded Options Valuation: Some Remarks in a Fuzzy Logic Framework

Esposito M.[a]; Orlando A.[b],*

[a] Istituto di Calcolo e Reti ad Alte Prestazioni (ICAR), CNR Napoli, Italy.
[b] Istituto per le Applicazioni del Calcolo 'Mauro Picone' (IAC), CNR Napoli, Italy.
*Corresponding author.

Received 16 May 2014; accepted 22 July 2014

Abstract

Variable annuities are investment products offered by Insurance Companies. They allow investors to place assets in mutual funds under the umbrella of a tax-deferred account. The account value of variable annuities fluctuates based on the performance of the selected mutual funds and therefore some risks are involved.

Recently there has been a growing interest in using fuzzy numbers to deal with financial uncertainty. Many authors have tried to deal fuzziness along with randomness in option pricing models.

Aim of the present contribution is to deepen the topic of evaluating the options embedded in variable annuities contracts in a fuzzy logic framework.

Key words: Variable annuities; Option pricing; Fuzzy logic; Cox Ross Rubenstein model

Esposito, M. & Orlando, A. (2014). Variable Annuities and Embedded Options Valuation: Some Remarks in a Fuzzy Logic Framework. *International Business and Management, 9*(1), 14-19. Available from: http://www.cscanada.net/index.php/ibm/article/view/5371
DOI: http://dx.doi.org/10.3968/5371

INTRODUCTION

Insurance markets around the world are changing because the investors are becoming more aware of investment opportunities outside the insurance sector. Policyholders want to enjoy the benefits of equity investment in conjunction with mortality protection and insurers around the world have developed new insurance products to meet this challenge.

Among the proposed innovative product we can find the Variable Annuities (VA). According to the National Association of Variable Annuity Writers (NAVA) "with a VA contract owners are able to choose from a wide range of investment options called sub-accounts, enabling them to direct some assets into investment funds that can help keep pace with inflation and some into more conservative choices. Sub-accounts are similar to mutual funds that are sold directly to the public in that they invest in stocks, bonds and money market portfolios". The VA products include guarantees, available as a rider feature of the overall product. Traditionally the guarantees were offered as a rider feature to the overall product package, but since 2000 insurance companies began offering more innovative guarantees, for an explicit price, as an optional choice to the customer.

VA have existed in USA since 1950s. NAVA report that the first variable annuity was issued in 1952. VA are now also spreading across Europe. Some of the more significant and high profile launches have been AXA's in France, Germany, Spain, Italy and Belgium as well as ING's launches in Spain, Hungary and Poland. Generali's launch (December 2007) in Italy and Ergo's launch (February 2008) launch in Germany. This is in addition to the various launches by Aegon, Hartford, Metlife and Lincoln in the U.K.

Over the years, many practical and academic contributions are dealing with the VAs and the guarantees embedded.

Recently, the academic literature has shown a fervent interest to the topic of VA (Bauer, Kling, & Russ, 2006), (Coleman, Li, & Patron, 2006), (Milevsky & Posner, 2001), (Milevsky & Promislow, 2001), (Milevsky & Salisbury, 2006).

The recent literature shows a great interest in applying fuzzy logic to insurance field. De Wit (1982) first applied fuzzy logic to insurance. Shapiro gives an interesting and exhaustive overview of insurance uses of fuzzy logic (Shapiro, 1998).

Aim of this paper is to focus on the pricing of options embedded in the VA contract resorting to fuzzy theory.

As well known the Black-Scholes model and the Cox-Ross-Rubinstein (CRR) model has been widely applied for computing the optimal warrant price. Referring to the results obtained by Li and Han (2009) we apply fuzzy set theory to the binomial tree option pricing model (CRR) to price the put option embedded in a GMDB guarantee of a variable annuity contract.

Taking the Knightian uncertainty of financial markets into consideration, the randomness and fuzziness of underlying should be evaluated by both probabilistic and fuzzy expectation. Han and Li make use of parabolic fuzzy numbers to discuss Han use the fuzzy binomial option pricing model with uncertainty of both randomness and fuzziness, and derive expression for the fuzzy risk neutral probabilities, along with fuzzy expression for option prices. Consequently, they obtain weighted intervals for the risk neutral probabilities and for the expected fuzzy option prices.

1. VARIABLE ANNUITY CONTRACT WITH A GUARANTEED MINIMUM DEATH BENEFIT

Variable annuities are investment products offered by Insurance Companies. The typical Variable Annuity (VA) is a unit linked annuity contract, which is normally purchased by a single premium payment. This kind of products allows investors to place assets in mutual funds under the umbrella of a tax-deferred account. The account value of variable annuities fluctuates based on the performance of the selected mutual funds and therefore some risks are involved.

The VA typically contains some embedded guarantees. The guarantees offered generally fall into four classes: Guaranteed Minimum Death Benefits (GMDBs) that guarantee a return of the principal invested upon the death of the policyholder; Guaranteed Minimum Accumulation Benefits (GMABs) similar to GMDBs except that instead of the guarantees being contingent on the death of the insured, they typically bite on specified policy anniversaries or between specified dates if the policy is still in-force. If the guarantee is available at maturity, they are called Guaranteed Minimum Maturity Benefits (GMMBs); Guaranteed Income Benefits (GMIBs) guarantee a minimum income stream (typically in the form of a life annuity) from a specified future point in time; Guaranteed Minimum Withdrawal Benefits (GMWBs) guarantee e minimum income stream trough regular withdrawals from the account balance.

Let us consider a portfolio of VA contracts offering GMDB guarantees and issued to C independent lives aged x being ω the ultimate age.

Each insured pays a unique premium P and at time zero the Company receives the sum $C*P$.

Assuming that the insured pays an initial charge for general expenses computed as a percentage c of the premium, the Company invests the net premiums $L_0^{\text{GMDB}} = \sum_{t=1}^{T} E[N_D(t)] \cdot E[\text{Max}[0, G_t - F_t]] \cdot \mathrm{e}^{-rt}$ into a Fund and each insured can choose between different investment strategies. By virtue of the GMDB guarantee, if the insured does not survive at the end of the month t, the Company pays a sum equal to the maximum between the guaranteed and the fund value.

The obligations the Company has to front for the GMDB at time $t \in 1,2,...,\varpi - x$ are:

$$\text{GMDB}_t = ND(t) \cdot \text{Max}[F_t, G_t] \quad (2.1)$$

Being $N_D(t)$ the number of deaths in $[t-1, t]$.

We assume that: $\{N_D(t)\}_{t=1}^{\varpi - x}$ is multinomial with parameters $\left(C; q_x,\ {}_{1|}q_x, \ldots, {}_{\varpi-1-x|}q_x,\right)$ being C the number of policies issued at time zero, ${}_{t|}q_x$ the probability that a life aged x dies in the (t+1)-th month after issue.

We assume that the guaranteed is computed according to a roll up guarantee that is the minimum benefit is equal to the single premium compounded with a constant interest rate (the roll up rate) and $G_t = P' \cdot e^{g \cdot t}$ with $t \in 1,2,...,\varpi - x$ and g the monthly guaranteed rate.

As well known by means of the put decomposition principle, (2.1) can be rewritten as follows:

$$\text{GMDB}_t = N_D(t) \cdot (F_t + \text{Max}[0, G_t - F_t]) \quad (2.2)$$

with $t \in [1,2,...,\varpi - x]$

(2.2) implies that upon death of the policyholder the Insurance Company pays the accumulated fund F_t plus an additional payoff of a put option with increasing strike price equal to G_t with $t \in 1,2,...,\varpi - x$.

If the Fund performs so poorly that the account value is below the guaranteed value, when the deaths occur the Company pays the difference.

Therefore, referring to a GMDB option, the Company's obligations month by month are:

$$L_t^{\text{GMDB}} = ND(t) \cdot \text{Max}[0, G_t - F_t]\ t \in [1,2,...,\varpi - x] \quad (2.3)$$

Our interest now is in the evaluation of the loss function L_t^{GMDB} at time zero. On the basis of the preceding considerations we need to price the portfolio of the embedded put options. These options are characterized by an increasing strike price and a stochastic maturity date, depending on the time of death of the insured.

In order to estimate the options we refer to a Binomial Tree option pricing model in a fuzzy logic framework.

2. OPTION PRICING AND UNCERTAINTY

As stressed in the previous section, we are interest in assessing the liabilities connected to the guarantees that the Company offers in the contract. Of course, we have to price the connected embedded options dealing with uncertainty that characterizes financial markets.

In asset pricing theory, uncertainty is modelled by means of state variables which play the role of sufficient statistics for the state of the world. The probability distributions, as well as the dynamic processes followed by the state variables, are assumed to be given and revealed to the agents in the economy.

Unfortunately in the real world distributions and stochastic dynamics are unknown or only partially known, and agent struggle to come by some hint about them. Usually this concept is referred to as information ambiguity, vagueness or uncertainty. Knight (Knight, 1921) stressed that the distinction between risk (a situation in which the relative odds of the events are known) and uncertainty (a situation in which no such probability assignment can be done) was a key feature to explain investment decisions. We refer to such uncertainty as Knigthian uncertainty. Classical probability theory is incapable of accounting for this type of uncertainty.

Li and Han (2009) provide a fuzzy binomial model of option price determination in which the Knightan uncertainty plays a role. By modelling the underlying in each state of the world as a fuzzy number, they obtain a possibility distribution on the risk neutral probability, i.e. a weighted interval of probability. By computing the option price under this measure, they get a weighted expected value interval for the price and thus they are able to determine a ‘most likely’ option value within the interval. Moreover, by means of the so-called defuzzification procedure it is possible to associate to the option price a crisp number that summarizes all the information contained. They get an index of the fuzziness present in the option price, that tells us the degree of imprecision intrinsic in the model.

The information given by this kind of approach can be very useful to the Company’s valuations, when pricing the options embedded into the contract to asses potential losses connected to the portfolio.

3. THE FUZZY GMDB OPTION PRICING

The VA contract offering a GMDB guarantee gives rise to the monthly Company obligations:

$L_t^{\text{GMDB}}=N_D(t)\cdot\text{Max}[0,G_t\text{-}F_t]$, $t\in[1,2,...,\varpi-x]$ (3.3)

Being t the random date of death and $N_D(t)$ the number of deaths at the random date t among the C insured lives at the inception of the contract.

The value at time zero of the loss function L_t^{GMDB}, for each t, is:

$$L_0^{\text{GMDB}}=\sum_{t=1}^{T}E[N_D(t)]\cdot E[\text{Max}[0,G_t-F_t]]\cdot \mathrm{e}^{-rt} \quad (3.4)$$

Being r the monthly risk free rate.

We assume that: $\{N_D(t)\}_{t=1}^{\varpi-x}$ is multinomial with parameters $\left(C;q_x,\ {}_{1|}q_x,\dots,{}_{\varpi\text{-}1\text{-}x|}q_x,\right)$ being C the number of policies issued at time zero, ${}_{t|}q_x$ the probability that a life aged x dies in the (t+1)-th month after issue. It follows that in (3.4) $E[N_D(t)]=C\cdot{}_{t|}q_x$.

Moreover, we assume that the demographic and financial factors are independent.

Therefore, equation (3.4) can be rewritten as follows:

$$\begin{aligned} L_0^{\text{GMDB}} &= \sum_{t=1}^{T} C\cdot{}_{t|}q_x\cdot E[\text{Max}[0,G_t-F_t]]\cdot \mathrm{e}^{-rt} \\ &= \sum_{t=1}^{T} C\cdot{}_{t|}q_x\cdot P_0(t) \end{aligned} \quad (3.5)$$

where $P_0(t)$ is the price, at time zero, of a put option with maturity t, underlying F_t and exercise price $G(t)=P'\cdot e^{g\cdot t}$.

Thus L_0^{GMDB} is a weighted average of ω-x European put options, where the weights are the expected number of deaths.

We need now to price the put embedded put options $P_0(t)$ for each t in a fuzzy framework. To this aim we refer to the results obtained by Li and Han (2009).

Let us know recall briefly the traditional Binomial option pricing model proposed by Cox, Ross and Rubinstein (CRR) in 1979. CRR model has a simple structure and it is widely applied in the financial market and is one of the basic options pricing methods.

The binomial pricing model traces the evolution of the option’s key underlying variables in discrete-time. This is done by means of a binomial lattice (tree), for a number of time steps between the valuation and expiration date. Each node in the lattice represents a possible price of the underlying at a given point in time.

Valuation is performed iteratively, starting at each of the final nodes (those that may be reached at the time of expiration), and then working backwards through the tree towards the first node (valuation date). The value computed at each stage is the value of the option at that point in time.

Option valuation using this method is, as described, a three-step process: price tree generation, calculation of option value at each final node, sequential calculation of the option value at each preceding node.

The tree of prices is produced by working forward from valuation date to expiration. At each step, it is assumed that the underlying instrument will move up or down by a specific factor u or d respectively (where, by definition, $u\geq 1$ and $0<d\leq 1$). So, if S_0 is the current price, then in the next period the price will either be.

$S_{\text{up}}=S_0\cdot u$ or $S_{\text{down}}=S_0\cdot d$. The up and down factors are calculated using the underlying volatility, σ, and the time duration of a step, Δt, measured in months, in our case (using the day count convention of the underlying instrument). From the condition that the variance of the log of the price is $\sigma^2\Delta t$, we have:

$$u = \mathrm{e}^{\sigma\sqrt{\Delta t}} \tag{3.1}$$

$$\mathrm{d} = \mathrm{e}^{-\sigma\sqrt{\Delta t}} = \frac{1}{u} \tag{3.2}$$

At each final node of the tree – i.e. at expiration of the option T – the option value is simply its intrinsic value that is Max [$(S_T - K)$, 0] for a call option and Max [$(K - S_T)$, 0], for a put option where K is the strike price and S_T is the spot price of the underlying asset at expiration time T. Once the above step is complete, the option value is then found for each node, starting at the penultimate time step, and working back to the first node of the tree (the valuation date) where the calculated result is the value of the option.

Let us now refer to the evolution of the Fund value referring to GMDB guarantee. Let us consider the date t=1.

The Fund value F_1 is given by either $F_0 d$ or $F_0 u$.

In a fuzzy framework, the up and down parameters u and d are parabolic fuzzy numbers denoted by $u = (u_1, u_2, u_3, u_4)_n$, $d = (d_1, d_2, d_3, d_4)_n$.

The α-cut of u and d is

$$\begin{aligned} u(\alpha) &= \left[\underline{u}(\alpha)\ \overline{u}(\alpha)\right] \\ &= \left[u_1 + \alpha^{1/n}(u_2 - u_1)\ u_4 - \alpha^{1/n}(u_4 - u_3)\right],\ \forall\alpha\in[0,1] \\ d(\alpha) &= \left[\underline{d}(\alpha)\ \overline{d}(\alpha)\right] \\ &= \left[d_1 + \alpha^{1/n}(d_2 - d_1)\ d_4 - \alpha^{1/n}(d_4 - d_3)\right],\ \forall\alpha\in[0,1] \end{aligned},$$

Being u and d parabolic fuzzy numbers, the fund value F_1 at t=1 in each state is represented by a parabolic fuzzy number too.

We denote the put payoff in state 'up' with *P(u)* and in state down with *P(d)*.

Applying the algebra of fuzzy numbers and remembering that in our case the strike price is given by $G(1) = P' e^{g \cdot t}$, we obtain the put payoff is equal to

$$\begin{aligned} P_u(\alpha) &= [\underline{P_u}(\alpha), \overline{P_u}(\alpha)] \\ &= [\mathrm{Max}(G(1) - F_0\underline{u}(\alpha), 0), \mathrm{Max}(G(1) - F_0\overline{u}(\alpha), 0)] \end{aligned}$$

Based on Li and Han results we get:

$$\begin{aligned} P_u(\alpha) &= [\underline{P_u}(\alpha), \overline{P_u}(\alpha)] \\ &= [G(1) - F_0(u_1 + \alpha^{1\backslash n}(u_2 - u_1)), \\ &\quad G(1) - F_0(u_4 - \alpha^{1\backslash n}(u_4 - u_3))] \end{aligned}$$

It is now possible to determine the put price P_0 by means of the risk-neutral valuation approach, as follows:

$$P_0 = \frac{1}{1+r}\hat{E}[P_1] = \frac{1}{1+r}[p_d P_d + p_u P_u]$$

Where $\hat{E}$ stands for expectation under the risk-neutral probabilities and P_1 is the payoff of the put at t=1.

Differently from the standard binomial option pricing model, it is possible to obtain risk-neutral probability intervals instead of point values. This is clearly a consequence of the assumptions on the stock price.

The risk-neutral probability intervals arise from the ambiguity of the stock price at time t=1. Moreover, the intervals of risk neutral probabilities are weighted, i.e. they are fuzzy numbers.

This is a very important feature of pricing options in a fuzzy framework, since it allows finding a weighted expected value interval for the option price.

$$\overline{p}_u(\alpha) = \frac{1 + r - d_1 - \alpha^{1\backslash n}(d_2 - d_1)}{u_1 - d_1 + \alpha^{1\backslash n}(u_2 + d_1 - u_1 - d_2)}$$

$$\underline{p}_d(\alpha) = \frac{-1 - r + u_1 + \alpha^{1\backslash n}(u_2 - u_1)}{u_1 - d_1 + \alpha^{1\backslash n}(u_2 + d_1 - u_1 - d_2)}$$

Solving system 5.5 we get:

$$\underline{p}_u(\alpha) = \frac{1 + r - d_4 - \alpha^{1\backslash n}(d_4 - d_3)}{u_4 - d_4 + \alpha^{1\backslash n}(u_4 - u_3 - d_4 + d_3)}$$

$$\overline{p}_d(\alpha) = \frac{-1 - r + u_4 - \alpha^{1\backslash n}(u_4 - u_3)}{u_4 - d_4 - \alpha^{1\backslash n}(u_4 - u_3 - d_4 + d_3)}$$

The two solutions represent the α-cut of the risk-neutral probability p_u and p_d:

$$p_u(\alpha) = [\underline{p}_u(\alpha), \overline{p_u}(\alpha)] \quad p_d(\alpha) = [\underline{p}_d(\alpha), \overline{p_d}(\alpha)].$$

Than the α-cut of P_0 is

$$P_0(\alpha) = [\underline{P_0}(\alpha), \overline{P_0}(\alpha)]$$

$$= \left[\begin{array}{c} \dfrac{G(1) - F_0(u_1 + \alpha^{1\backslash n}(u_2 - u_1))}{1+r}\dfrac{1 + r - d_4 + \alpha^{1\backslash n}(d_4 - d_3)}{u_4 - d_4 - \alpha^{1\backslash n}(u_4 - u_3 - d_4 + d_3)}, \\ \dfrac{G(1) - F_0(u_4 + \alpha^{1\backslash n}(u_4 - u_3))}{1+r}\dfrac{1 + r - d_1 + \alpha^{1\backslash n}(d_2 - d_1)}{u_1 - d_1 - \alpha^{1\backslash n}(u_2 + u_1 - u_1 - d_2)} \end{array}\right]$$

It is easy to prove that α increases the put option interval of prices shrinks. If $u_2 = u_3, d_2 = d_3$ and α=1 the interval collapses into one single value.

Extending the methodology to a multiple period, we get:

$$P_0(t)=\left(\frac{1}{1+r}\right)^t\left[\sum_{i=0}^{t}\binom{t}{i}\underline{p}_u^i(\alpha)\underline{p}_d^{t-i}(\alpha)\underline{P}_{t,i}(\alpha)\ \sum_{i=0}^{t}\binom{t}{i}\overline{p}_u^i(\alpha)\overline{p}_d^{t-i}(\alpha)\overline{P}_{t,i}(\alpha)\right]$$

Where:

$$\begin{aligned}P_{t,i}(\alpha)&=[\underline{P_{t,i}}(\alpha),\overline{P_{t,i}}(\alpha)]\\&=[\mathrm{Max}(G(t)-(F_0u_1^id_1^{t-i}+\alpha^{1\backslash n}F_0(u_2^id_2^{t-i}-u_1^id_1^{t-i})),0),\\&\quad\mathrm{Max}(G(t)-(F_0u_4^id_4^{t-i}+\alpha^{1\backslash n}F_0(u_4^id_4^{t-i}-u_3^id_3^{t-i}),0)]\end{aligned}$$

CONCLUSION AND FUTURE DIRECTIONS

We are interested in pricing variable annuity guarantees of the typical VA contract described in section 2.

In the recent literature two stochastic approaches are implemented: The traditional actuarial approach which uses a ‘real world’ projection and the market consistent approach which typically uses a ‘risk neutral’ projection. In general, pricing practice varies across different countries and companies.

We believe that the use of market consistent approach for pricing variable annuities guarantees is the most appropriate method for actuaries and companies today. This approach uses stochastic valuation techniques consistent with the pricing of options. This flexible methodology enables most product benefit and charging structures to be accommodated, and facilitates the calculation of risk exposures that can be used to construct and manage a dynamic hedge portfolio.

As well known the Black-Scholes model and the Cox-Ross- Rubinstein (CRR) model has been widely applied for computing the optimal warrant price and are typically used in insurance to price embedded options.

In asset pricing theory, uncertainty is modelled by means of state variables which play the role of sufficient statistics for the state of the world. The probability distributions, as well as the dynamic processes followed by the state variables, are assumed to be given and revealed to the agents in the economy.

Unfortunately in the real world distributions and stochastic dynamics are unknown or only partially known, and agent struggle to come by some hint about them. Usually this concept is referred to as information ambiguity, vagueness or uncertainty. Knight (1921) stressed that the distinction between risk (a situation in which the relative odds of the events are known) and uncertainty (a situation in which no such probability assignment can be done) was a key feature to explain investment decisions. We refer to such uncertainty as Knigthian uncertainty. Classical probability theory is incapable of accounting for this type of uncertainty.

Li and Han (2009) provide a fuzzy binomial model of option price determination in which the Knightan uncertainty plays a role. By modelling the underlying in each state of the world as a fuzzy number, they obtain a possibility distribution on the risk neutral probability, i.e. a weighted interval of probability. By computing the option price under this measure, they get a weighted expected value interval for the price and thus they are able to determine a ‘most likely’ option value within the interval. Moreover, by means of the so-called defuzzification procedure it is possible to associate to the option price a crisp number that summarizes all the information contained. They get an index of the fuzziness present in the option price, that tells us the degree of imprecision intrinsic in the model.

The information given by this kind of approach can be very useful to the company’s valuations, when pricing the options embedded into the contract.

Of course the topic of setting what price should the policy holder be charged for guarantee benefit is an important issue for actuaries and risk managers.

Moreover, a suitable pricing technique is essential to asses potential losses connected to the portfolio.

On the other hand, the complex hybrid equity and interest rate options embedded in variable annuity products present formidable hedging challenges for the insurers who write them.

Actuarial risks of policyholders’ behaviour complicate this problem further. Few insurers have developed complete liability valuation models integrating all these factors. Yet, growth in the VA markets requires not only comprehensive valuation models, but also efficient methods to measure the prospective performance of different hedging programs around these risks, and a way to help insurers decide how they are going to hedge. Aim of this research is to deepen these issues in a fuzzy logic framework.

REFERENCES

Appadoo, S. S., & Thavaneswaran, A. (2013). Recent developments in fuzzy sets approach in option pricing. *Journal of Mathematical Finance*, *3*(2), *312-322.*

Bauer, D., Kling, A., & Russ, J. (2006). A universal pricing framework for guaranteed minimum benefits in variable annuities. *Astin Bulletin*, *38*(2), 621-651.

Coleman, T. F., Li, Y., & Patron, M. (2006). Hedging guarantees in variable annuities under both equity and interest rate risks. *Insurance: Mathematics and Economics*, *38*, 215-228.

Cox, J. C., Ross, S., & Rubinstein, M. (1979). Option pricing, a simplified approach. *Journal of Financial Economics*, *7*(3), 229-263.

Cox, E. (1994). *The fuzzy system handbook*. New York: Academic Press.

De Wit G. W. (1982). Underwriting and uncertainty. *Insurance: Mathematics and Economics*, *1*, 277-285.

Knight, F. H. (1921). *Risk, uncertainty and profit Boston, MA*. Hart, Schaffner & Marx Houghton Mifflin Company.

Li, W., & Han, L. Y. (2009). *The fuzzy binomial option pricing model under Knightian uncertainty* (pp. 399-403). Proceedings of sixth international conference on fuzzy systems and knowledge discovery, Tianjin, China, 14-16 August. IEEE.

Liu, S. X., Chen, Y., & Xu, N. (2009). Application of fuzzy theory to binomial option pricing model. *Fuzzy Information and Engineering* (pp. 63-70). Berlin: Springer-Verlag.

Milevsky, M., & Posner, S. E. (2001). The titanic option: Valuation of the guaranteed minimum death benefit in variable annuities and mutual funds. *The journal of Risk and Insurance*, *68*(1)*,* 91-126.

Milevsky, M. A., & Promislow, S. D. (2001). Mortality derivatives and the option to annuities. *Insurance: Mathematics and Economics*, *29*, 299-318.

Milevsky, M. A., & Salisbury, T. S. (2006). Financial valuation of guaranteed minimum withdrawal benefits. *Insurance: Mathematics and Economics*, *38*, *21-38*.

Shapiro, A. F. (2007). An overview of insurance uses of fuzzy logic. In Wang P. P., & Kuo, T. W. (Eds), *Computational and Intelligence in Economics and Finance* (II, pp. 25-63). Springer.