CSCanada
Advances in Natural Science
Vol. 9, No. 1, 2016, pp. 1-3
DOI:10.3968/8262

ISSN 1715-7862 [PRINT]
ISSN 1715-7870 [ONLINE]
www.cscanada.net
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Sharp Bounds for Spectral Radius of Graphs Presented by *K*-Neighbour of the Vertices

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Received 24 December 2015; accepted 10 February 2016
Published online 26 February 2016

Abstract

Let $G = (V, E)$ be a simple connected graph with $V(G) = \{v_1, v_2, \cdots, v_n\}$ and degree sequence $d_1, d_2, \cdots, d_n$. Denote $t_k(i) = \sum_{(v_i,v_j)\in E} t_{k-1}(j), m_k(i) = \frac{t_{k+1}(i)}{t_k(i)}$, where k is a positive integer number and $v_i \in V(G)$ and note that $t_0(i)=d_i$. Let $\rho(G)$ be the largest eigenvalue of adjacent matrix of G. In this paper, we present sharp upper and lower bounds of $\rho(G)$ in terms of $m_k(i)$ (see theorem (2.1)). From which, we can obtain some known results, and our result is better than other results in some case.

Key words: Spectral radius; Bound; *k*-neighbour

Li, P. (2016). Sharp Bounds for Spectral Radius of Graphs Presented by K-Neighbour of the Vertices. *Advances in Natural Science*, *9*(1), 1-3. Available from: http://www.cscanada.net/index.php/ans/article/view/8262
DOI: http://dx.doi.org/10.3968/8262

INTRODUCTION

Let $G = (V, E)$ be a connected graph without loops and multiedges and vertex set $V=\{v_1, v_2, \cdots, v_n\}$. The degree d_i of a vertex v_i in the graph G is defined to be the number of edges in G adjacent to v_i. For $v_i \in V(G)$, $N(v_i)$ denotes the neighbors of v_i. The 2-degree of v_i (Brualdi & Hoffman, 1985) is the sum of the degrees of the vertices adjacent to v_i and denoted by t_i ,and the average-degree of v_i is $m_i = \frac{t_i}{d_i}$. Here we define

$$t_k(i) = \sum_{(v_i,v_j)\in E} t_{k-1}(j), m_k(i) = \frac{t_{k+1}(i)}{t_k(i)},$$

where k is a positive integer number. Note that

$$t_0(i)=d_i,\ m_0(i)=m_i.$$

Let $A(G) = (a_{ij})$, $a_{ij}=1$ if $(v_i,v_j)\in E$ and $a_{ij}=0$ otherwise be the adjacency matrix of G . It follows immediately that if G is a simple graph, then $A(G)$ is a symmetric (0, 1) matrix in which every diagonal entry is zero. Since $A(G)$ is real and symmetric, its eigenvalues are real. The spectral radius of G, denoted by $\rho(G)$, is the largest eigenvalue of $A(G)$. Note that if G is connected, then $A(G)$ is irreducible, and so by the PerronFrobenius theory of non-negative matrices, $\rho(G)$ has multiplicity one and there exists a unique positive unit eigenvector (also called Perron-eigenvector) corresponding to $\rho(G)$.

Up to now, many bounds for $\rho(G)$ were given .For example, Kinkar Ch.Das and Pawan Kumar (Das & Kumar, 2004) gave a bound of spectral radius for graphs:

$$\min\{\sqrt{m_i m_j} : ij \in E\} \le \rho(G) \le \max\{\sqrt{m_i m_j} : ij \in E\}, \qquad (1)$$

where m_i is the average degree of v_i. Moreover, the equality holds if and only if G is either a graph with all the vertices of equal average degree or a bipartite graph with vertices of same set having equal average degree.

In this paper, we will generalize the Kinkar Ch.Das and Pawan Kumar`s bound and obtain the upper and lower bounds on $\rho(G)$ in terms of $m_k(i)$. From which, we can obtain some known results (for example (1)). We will give an example to show that our result is better than the bound (1) in some case.

Now we introduce some lemmas which will be used later on.

Lemma 1.1 (Horn & Johnson,1985). *Let A be a nonnegative matrix of order n. R_i be the ith row sum of A. Then*

$$\min\{R_i : 1 \le i \le n\} \le \rho(A) \le \max\{R_i : 1 \le i \le n\}.$$

If A is irreducible, then each equality holds if and only *if* $R_1=R_2=\cdots=R_n$.

Lemma 1.2 *Let G be a bipartite graph with bipartition* $V = U \cup W$ *and* $m_k(i)=\alpha$ *for* $v_i \in U$, $m_k(j)=\beta$ *for* $v_j \in W$, *Then,* $\rho(G)=\sqrt{\alpha\beta}$.

Proof. Let A be the adjacency matrix of G. Obviously, $\rho(G)$ is the spectral radius of the matrix $M=K^{-1}(D^{-1}AD)K$ too, where D=diag $t_k(1),t_k(2),\cdots, t_k(n)$, K =diag(k_1,k_2,k_n), $k_i=\sqrt{\alpha}$ when $v_i \in U$ and $k_i=\sqrt{\beta}$ when $v_i \in W$, $1 \le i \le n$. Then (i, j)th element of the matrix M is equal to

$$\begin{cases} \sqrt{\dfrac{\beta}{\alpha}}\dfrac{t_k(j)}{t_k(i)} & if\,(v_i,v_j\in E), v_i\in U; \\ \sqrt{\dfrac{\alpha}{\beta}}\dfrac{t_k(j)}{t_k(i)} & if\,(v_i,v_j\in E), v_i\in W; \\ 0 & otherwise. \end{cases}$$

So each row sum of the matrix M is equal to $\sqrt{\alpha\beta}$. Thus, by Lemma 1.1, we have $\rho(G)=\sqrt{\alpha\beta}$.

THE BOUNDS OF SPECTRAL RADIUS

Theorem 2.1 *Let G be a connected graph. Then*

$$\min\{\sqrt{m_k(i)m_k(j)}:(v_i,v_j)\in E\} \le \rho(G) \le \max\{\sqrt{m_k(i)m_k(j)}:(v_i,v_j)\in E\}. \quad (2)$$

Moveover, either of equality holds for a particular value of k if and only if $m_k(1)=m_k(2)=\cdots m_k(n)$ or G is a bipartite graph with the partition $\{v_1,v_2,\cdots,v_{n_1}\}\cup\{v_{n_1+1},\cdots,v_n\}$ and $m_k(1)=m_k(2)=\cdots m_k(n_1)$, $m_k(n_1+1)=\cdots m_k(n)$.

Proof. It is easy to see that the proof of lower bound is similar as the upper bound, so we only give the proof of upper bound. Let D = diag $t_k(1),t_k(2),\cdots,t_k(n)$. Obviously, $D^{-1}AD$ and A have the same spectral radius. Let $X=(x_1, x_2,\cdots, x_n)^T$ be an eigenvector of $D^{-1}AD$ corresponding to the spectral radius $\rho(G)$. Let one eigencomponent (say x_1) be equal to 1 and the other eigencomponents be less than or equal to 1, that is, $x_1 = 1$, and $0 < x_k \le 1$ for all k. Let $x_2=\max\{x_k:(v_1,v_k)\in E\}\ge\max\{x_k:(v_i, v_k)\in E\}$ when $x_i = 1$. Now the (i, j)th element of $D^{-1}AD$ is $\dfrac{t_k(j)}{t_k(i)}$, if $(v_i, v_j)\in E$, and 0 otherwise.

We have

$$D^{-1}ADX=\rho(G)X. \quad (3)$$

From the first equation of (3), we have

$$\rho x_1=\sum_{(v_1,v_j)\in E}\frac{t_k(j)x_k}{t_k(1)},$$

$$\rho\le m_k(1)x_2. \quad (4)$$

From the second equation of (3), we have

$$\rho x_2=\sum_{(v_2,v_j)\in E}\frac{t_k(j)x_k}{t_k(2)},$$

$$\rho x_2\le m_k(2). \quad (5)$$

From (4) and (5), we get

$$\rho^2\le m_k(1)m_k(2).$$

Therefore,

$$\rho^2\le\sqrt{m_k(1)m_k(2)}.$$

Hence,

$$\rho(G)\le\max\{\sqrt{m_k(i)m_k(j)}:(v_i,v_j)\in E\}.$$

Now suppose that the equality in (2) holds. Then all inequalities in the above argument must be equalities. In particular, we have from (4) that $x_j=x_2$ for all k, $(v_i, v_j)\in E$, also from (5) that $x_j=x_1=1$ for all k, $(v_2, v_j)\in E$. Now we distinguish two cases bellow:

Case (i): $x_2=1$. Let $V_1=\{k:x_k=1\}$. If $V_1\ne V$, there exist vertices $r,p\in V_1, q\notin V_1$ such that $(v_r, v_p)\in E$, and $(v_p, v_q)\in E$. since G is connected, so $x_r=x_p=x_1=1$.

From the r-th equation of (3), we have

$$\rho x_r=\sum_{(v_r,v_j)\in E}\frac{t_k(j)x_j}{t_k(r)}\le m_k(r).$$

From the p-th equation of (3), we have

$$\rho x_p=\sum_{(v_p,v_j)\in E}\frac{t_k(j)x_j}{t_k(p)}< m_k(p).$$

So we have $\rho(G)<\sqrt{m_k(r)m_k(p)}$ which contradicts that the equality holds in (2). Thus $V_1=V$ and $m_k(1)=m_k(2)=\cdots m_k(n)=\rho$.

Case (ii): $x2<1$. We have $x_j=1$, $v_j\in N(v_2)$ and $v_j=x_2$, $v_j\in N(v_1)$. Let $U=\{k:x_k=1\}$and $W=\{k:x_k=x_2\}$, so $N(v_1)\subseteq W$, and $N(v_2)\subseteq U$. Further, for any vertex $v_r\in N(N(v_1))$, there exists a vertex $v_p\in N(v_1)$, such that $v_1v_p\in E$ and $v_pv_r\in E$, therefore $x_p=x_2$. From the p-th equation of (3), we have

$$\rho x_p=\sum_{(v_p,v_j)\in E}\frac{t_k(j)x_j}{t_k(p)}\le m_k(p),$$

Using (4), we get

$$\rho^2\le m_k(1)m_k(p),$$

since we have

$$\rho(G)=\max\{\sqrt{m_k(i)m_k(j)}:(v_i,v_j)\in E\}\ge\sqrt{m_k(1)m_k(p)},$$

so $\rho(G)<\sqrt{m_k(r)m_k(p)}$, which shows that $x_r=1$. hence $N(N(v_1))\subseteq U$. By a similar argument, we can show that $N(N(v_1))\subseteq W$. Continuing the procedure, it is easy to see, since G is connected, that $V = U \cup W$ and that the subgraphs induced by U and W, respectively, are empty graphs. Hence G is bipartite and $m_k(i)$ are the same for $v_i \in U$, $mk(j)$ are the same for $v_i \in W$.

Conversely, if G is a graph with $m_k(1)=m_k(2)=\cdots m_k(n)$, then the equality in (2) is satisfied. Let G be a bipartite graph with bipartition $V = U \cup W$ and $m_k(i)=\alpha$ for $v_i \in U$, $m_k(j)=\beta$ *for* $v_j \in W$. Then, by lemma 1.2,

$$\rho(G) = \sqrt{\alpha\beta} = \max\{\sqrt{m_k(i)m_k(j)} : (v_i, v_j) \in E\},$$

we complete the proof.

Note 2.2 *If* $k = 0$*, then the inequality* (2) *is the Kinkar Ch.Das and Pawan Kumar's bound* (1). *Here we give an example to show that* (2) *is better than the Kinkar Ch.Das and Pawan Kumar^s bound in some case. Let G be a graph shown in Figure 1. Then the bound* (2) *is* $2.52 \le \rho(G) \le 2.67$ *when k =1, and Kinkar Ch.Das and Pawan Kumar^s bound is* $2.49 \le \rho(G) \le 2.83$. *Thus in that case ,* (2) *is better than the Kinkar Ch.Das and Pawan Kumar^s bound.*

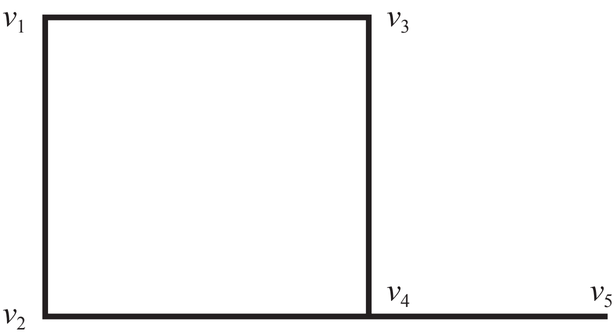

Figure 1
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Consequently, from (2) we have the following results.

Corollary 2.3 *Let G be a simple connected graph. Then*

$$\min\{m_k(i): i \in V\} \le \rho(G) \le \max\{m_k(i): i \in V\}. \quad (6)$$

Moveover, equality holds for a particular value of k if and only if $m_k(1)=m_k(2)=\cdots m_k(n)$.

Note 2.4 *If* $k = 0$*, then the inequality* (6) *is the Favaron et.al., s bound* (Favaron, Maheo, & Sacle, 1993, p.193).

Corollary 2.5 (a) *Let G be a graph with* $m_k(v)=p$. *For each* $v \in V(G)$*, then* $\rho(G)=p$.

(b) *Let G be a bipartite graph with the bipartition (X, Y). If* $m_k(v)=p_x$ *for each* $v \in X$ *and* $m_k(v)=p_y$. *For each* $v \in Y$*, then* $\rho(G) = \sqrt{p_x p_y}$.

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